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# OPTIMIZING COLLECTION CENTRE LOCATION ON A HEATMAP USING THE WEISZFELD ALGORITHM FOR THE FERMAT-WEBER PROBLEM

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***Abstract:*** *Facility location problems play an important role in logistics, particularly in the design of collection and distribution centers. This study presents a novel approach to determining the optimal location of a collection center based on existing collection tasks, represented by coordinates and associated quantities. The proposed method models the entire area using a heatmap, capturing the spatial distribution of goods to be collected. Based on this heatmap, the optimal location of the collection point is identified by solving the Fermat-Weber Problem (FWP) using the Weiszfeld Algorithm (WA), ensuring minimal materials handling costs. The study demonstrates how the heatmap representation enables a more flexible and dynamic facility location strategy compared to traditional point-based optimization approach. Numerical example shows that the locations derived from the heatmap closely approximate those computed directly from discrete collection points, making this approach suitable for collection tasks with dynamic and continuously changing demand patterns.*

***Keywords:*** *accelerate convergence, cost function, facility location, Fermat-Weber Problem, singularity issues, Weiszfeld algorithm.*

## 1. INTRODUCTION

The choice of optimal location for facilities such as warehouses, distribution centers, and collection points can significantly impact transportation costs, delivery times, and overall service quality. While there are complex methods for determining optimal facility locations, such as genetic algorithm [1], random search algorithm [2], deep reinforcement learning [3], path-relinking metaheuristic [4], iterated local search [5] or branch and bound [6], simpler approaches are also highly valuable. Simple methods offer the advantage of being easier to implement and requiring less computational power, making them suitable for smaller-scale operations or when rapid decision-making is necessary. For collection centers, the optimal site selection is crucial in minimizing transportation distances and ensuring timely service. These approaches provide a practical balance between accuracy and feasibility, especially in dynamic or resource-constrained environments.

This article focuses on the description of facility location of collection centers as a Fermat-Weber problem. The highlights of the article are the followings: (1) description of the Fermat-Weber problem in *n*-dimensional and 2-dimensional search space; (2) description of the Weiszfeld Algorithm to solve the Fermat-Weber Problem; (3) case study to demonstrate the potential of Weiszfeld algorithm to solve facility location problem of collection centers.

The article is organized as follows. Chapter 2 presents a short literature review, including both descriptive and content analyses Chapter 3 discusses the Fermat-Weber problems and the Weiszfeld algorithm. Chapter 4 presents a numerical study, which shows, that the proposed heatmap-based approach effectively captured the spatial distribution of collection

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tasks, offering a more comprehensive representation compared to conventional discrete point-based methods.

## 2. LITERATURE REVIEW

In this chapter, the research results in the field of facility location are summarized. This section includes a descriptive and a content analysis. Within the frame of the literature review, Scopus was used to identify the most important scientific results regarding reusable packaging materials. The review focuses on the highlight and main research directions of facility location focusing on collection centers.

Firstly, the relevant terms must be defined. In this first phase I have chosen a simply keyword: "facility location AND collection center" to find a wide range of articles to perform a descriptive analysis of articles. Initially, 454 articles were identified.

As Fig. 1 shows, "facility location problems" has been researched in the past 60 years. The first article in this field was published in 1962. The number of published papers focusing on different aspects of facility location problems has significantly increased in the last 20 years; it shows the importance of this research field.

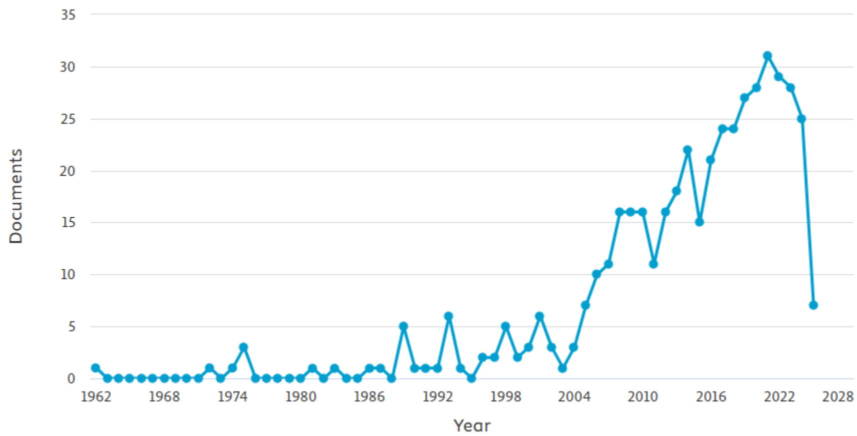


*Figure 1. Published articles per year in the field of facility location resulted by a Scopus search (Source: www.scopus.com)*

We can analyze the distribution of published articles per year and per source, as shown in Fig. 2.

It can be seen, that a wide range of articles in the field of facility location has been published in five scientific journals: Computers and Operations Research, Computers and Industrial Engineering, European Journal of Operational Researches, Journal of Cleaner Production and Acta Horticulture. The title and the main topic of these scientific journals shows, that facility location covers interdisciplinary sciences. The CiteScore is shown in Fig. 3.

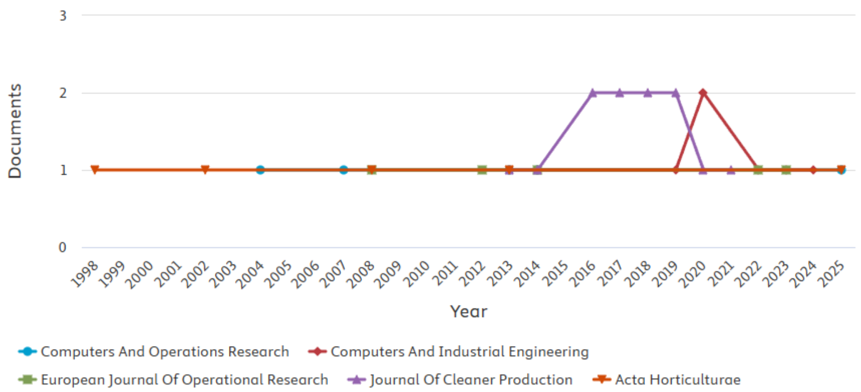


*Figure 2. Published articles per year per source in the field of facility location resulted by a Scopus search (Source: www.scopus.com)*

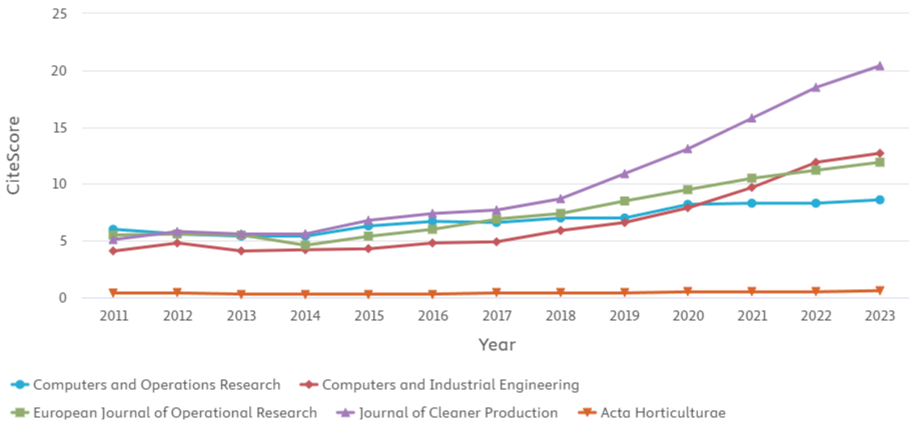


*Figure 3. CiteScore publication per year in the field of reusable packaging resulted by a Scopus search (Source: www.scopus.com)*

The analysis of the subject area of research works shows (see Fig. 4), that the most important engineering-related subject areas in the Scopus are the following: computer science, environmental science, mathematics, decision making and energy.

The next phase of the literature review is the content analysis, where the initial articles are filtered using additional keywords of Scopus. The location of facilities in supply chains and reverse logistics networks is an important decision that affects both efficiency and environmental sustainability. Recently, many methods have been suggested to solve the complex problem of finding the best location for waste collection and reverse logistics centers. These methods combine mathematical models, optimization techniques, and decision-making tools to create the best network design. One important contribution in this area comes from Amiri-Aref and Doostmohammadi [7], who study the design of multi-stage supply chain networks. They focus on hybrid retailer/collection center facilities, considering both long-term decisions, like where to locate the centers, and short-term decisions, like inventory management and fleet size. They use a mixed integer linear programming (MILP) model and solve it with a combination of Relax-and-Fix and Fix-and-Optimize methods.

Their tests show that this approach works well and finds nearly optimal solutions in a reasonable time, especially for smaller problems.

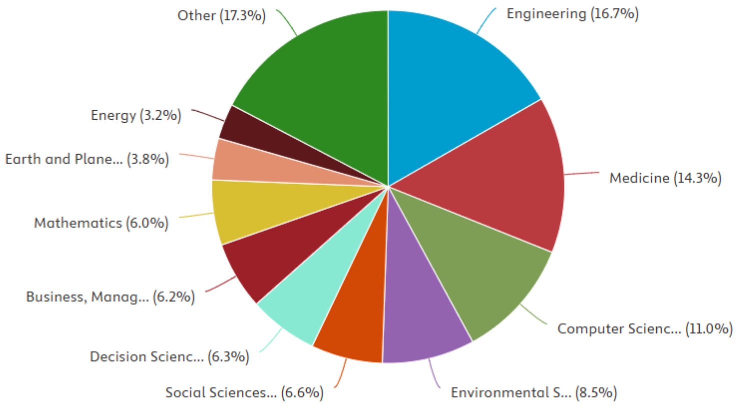


*Figure 4. Published articles by subject area related to facility location resulted by a Scopus search (Source: www.scopus.com)*

Ayyildiz and Erdogan [8] look at the problem of locating organic waste collection and recycling centers, which is particularly important in developing countries dealing with rapid urban growth and limited resources. They use the Analytical Hierarchy Process (AHP) and fuzzy interval type-2 sets (T2F) to weigh location factors and evaluate different sites using the Combinative Distance-based Assessment (CODAS) method. Their study finds the best location for an organic waste center in Trabzon, Turkey, providing a useful tool for waste management in the area. Yang and Chen [9] focus on the location problem for reverse logistics in closed-loop supply chains. They use a fuzzy analytic network process (FANP) and multi-objective mixed-integer linear programming (MOMILP) to consider both qualitative and quantitative factors like environmental impact, costs, and responsiveness. Their model allows decision-makers to look at different scenarios, making it flexible for uncertain situations. Caramia and Giordani [10] propose a two-level optimization model to decide where to locate different waste collection centers. In their model, a municipal firm (the leader) determines the locations and capacities of the centers, while users (the followers) decide whether to cooperate based on waste recycling plans. They use a randomized-rounding method to solve the problem and compare their results to existing approaches, showing their method's effectiveness.

Ocampo et al. [11] suggest a multi-criteria decision-making framework for mapping collection and distribution centers in reverse logistics. They use fuzzy DEMATEL, fuzzy ANP, and fuzzy AHP to evaluate factors like government policies, economic conditions, and remanufacturing uncertainties. Their study highlights the importance of considering multiple factors to improve decision-making, especially in complex reverse logistics systems.

Ferri et al. [12] explore the location of urban waste collection and inspection centers in reverse logistics. They propose a mathematical model to centralize waste collection and improve waste management. Their case study in São Mateus, Brazil, shows how selective collection and reverse logistics can reduce environmental impacts and improve urban waste management.

Tari and Alumur [13] examine the collection center location problem while considering fairness. They look at multiple objectives, like minimizing costs and ensuring fair distribution. Their model focuses on steady product flow and equal treatment of companies, with applications in the collection of waste electrical and electronic equipment in Turkey. These studies collectively highlight the importance of optimizing the location of collection centers within reverse logistics and CLSC networks. They provide valuable insights into how various decision-making models and optimization techniques can be applied to improve operational efficiency, reduce costs, and promote sustainability in waste management and reverse logistics systems. The integration of fuzzy logic, multi-criteria decision-making, and heuristic algorithms in solving complex location problems has proven effective across different sectors, including waste management, blood collection, and product remanufacturing. Compared to the approaches discussed in the literature review, the Fermat-Weber method using the Weiszfeld algorithm can be particularly beneficial in scenarios where the distances between collection centers or facilities are evenly distributed, and the goal is to minimize the overall distance to all other points. This method is well-suited for optimizing logistics networks by identifying a central location that reduces the travel distance, especially in cases where other complex factors, such as environmental impact or capacity constraints, are not as prominent. In contrast to more intricate models that consider a variety of dynamic and qualitative factors, the Fermat-Weber method provides a simpler, more direct approach to locating facilities when the main focus is on reducing transportation costs.

## 3. MATERIALS AND METHODS

The Fermat-Weber problem is a fundamental location optimization problem that is suitable to determine the optimal point $\boldsymbol{x}^{\boldsymbol{C}} \in \mathbb{R}^n$ in an n-dimensional search space that minimizes the weighted sum of Euclidean distances to a given set of points $\boldsymbol{x}_i \in \mathbb{R}^n$, where each point has an associated weight $w_i$.

This problem is widely used in facility location planning, logistics, and supply chain management to determine the best position for a distribution center, hub, or central facility.

The mathematical formulation of the problem is given by:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}) = \sum_{i=1}^{m} w_i \cdot \|\boldsymbol{x} - \boldsymbol{x}_i\|, \tag{1}$$

where the Euclidean distance between $\boldsymbol{x}$ and $\boldsymbol{x}_i$ is defined as:

$$\|\boldsymbol{x} - \boldsymbol{x}_i\| = \sqrt{\sum_{j=1}^{n} (x_j - x_{ij})^2}. \tag{2}$$

We consider the special case of $n = 2$, where the goal is to find a point $(x, y)$ in a two-dimensional plane that minimizes the weighted sum of Euclidean distances to a given set of points $(x_i, y_i)$. The problem is written as:

$$\min_{(x,y)} f(x, y) = \sum_{i=1}^{m} w_i \cdot \sqrt{(x - x_i)^2 + (y - y_i)^2}. \tag{3}$$

Since the function $f(\boldsymbol{x})$ is not differentiable at any of the given points $\boldsymbol{x}_i$, a direct analytical solution is difficult. Instead, iterative numerical methods, such as the Weiszfeld algorithm, are commonly used to approximate the optimal solution.

The Weiszfeld algorithm is an iterative procedure that refines the estimated location step by step. In the case of an n-dimensional search space, the algorithm starts from an initial estimate $\boldsymbol{x}^{(0)}$ typically chosen as the weighted centroid:

$$\boldsymbol{x}^{(0)} = \frac{\sum_{i=1}^{m} w_i \cdot \boldsymbol{x}_i}{\sum_{i=1}^{m} w_i}. \tag{4}$$

Then, the update step at each iteration is given by:

$$\boldsymbol{x}^{(k+1)} = \frac{\sum_{i=1}^{m} \frac{w_i \cdot \boldsymbol{x}_i}{d_i^{(k)}}}{\sum_{i=1}^{m} \frac{w_i}{d_i^{(k)}}}, \tag{5}$$

where

$$d_i^{(k)} = \left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}_i\right\| = \sqrt{\sum_{j=1}^{n} (x_j^{(k)} - x_{ij})^2}. \tag{6}$$

This iterative process continues until the termination criterion is met, typically when the change in $\boldsymbol{x}$ is below a predefined threshold $\varepsilon$.

$$\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| < \varepsilon. \tag{7}$$

In the case of a 2-dimensional search space (in a plane, or in a map), the algorithm starts from an initial estimate $\boldsymbol{x}^{(0)}$ typically chosen as the weighted centroid:

$$x^{(0)} = \frac{\sum_{i=1}^{m} w_i \cdot x_i}{\sum_{i=1}^{m} w_i}, \tag{8}$$

$$y^{(0)} = \frac{\sum_{i=1}^{m} w_i \cdot y_i}{\sum_{i=1}^{m} w_i}. \tag{9}$$

Then, the update step at each iteration is given by:

$$x^{(k+1)} = \frac{\sum_{i=1}^{m} \frac{w_i \cdot x_i}{d_i^{(k)}}}{\sum_{i=1}^{m} \frac{w_i}{d_i^{(k)}}}, \tag{10}$$

$$y^{(k+1)} = \frac{\sum_{i=1}^{m} \frac{w_i \cdot y_i}{d_i^{(k)}}}{\sum_{i=1}^{m} \frac{w_i}{d_i^{(k)}}}, \tag{11}$$

where

$$d_i^{(k)} = \sqrt{(x^{(k)} - x_i)^2 + (y^{(k)} - y_i)^2}. \tag{12}$$

A well-known issue in the Weiszfeld algorithm arises when the iterative solution $\boldsymbol{x}^{(k)}$ coincides with one of the given points $\boldsymbol{x}_i$, causing division by zero in the update formulas. To avoid this, a small perturbation $p$ is introduced, ensuring that the denominator does not become zero:

$$d_i^{(k)} = \max\left(\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}_i\right\|, p\right), \tag{13}$$

or in the case of the 2-dimensional search space:

$$d_i^{(k)} = \max\left(\sqrt{(x^{(k)} - x_i)^2 + (y^{(k)} - y_i)^2}, p\right). \tag{14}$$

It is possible to accelerate the convergence speed of the Weiszfeld algorithm using an acceleration factor $\mu$:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k+1)} + \mu \cdot \left(\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right), \tag{15}$$

or in the case of the 2-dimensional search space:

$$x^{(k+1)} = x^{(k+1)} + \mu \cdot \left(x^{(k+1)} - x^{(k)}\right), \tag{16}$$

$$y^{(k+1)} = y^{(k+1)} + \mu \cdot \left(y^{(k+1)} - y^{(k)}\right). \tag{17}$$

## 4. Results

Within the frame of this chapter, a case study illustrates how to determine the optimal location of the collection center, given the known positions of the pickup points and the quantities of goods collected in the past period. In this case study, the input parameters of the optimization problem are the locations of 10 pickup points and the number of pickup operation in the past period. Table I and Fig. 5 shows the locations of these pickup points and the collected quantities (number of pickup operations).

*Table I.*

*Locations of pickup points and the collected quantities (pickup operations).*

| Pickup point ID | X coordinate | Y coordinate | Number of pickup operations |
|---|---|---|---|
| **1** | 3003.0 | 1507.5 | 55 |
| **2** | 3130.0 | 1610.0 | 6 |
| **3** | 3300.0 | 1600.0 | 6 |
| **4** | 3154.5 | 1623.0 | 5 |
| **5** | 3200.0 | 1651.5 | 22 |
| **6** | 3301.5 | 1500.0 | 32 |
| **7** | 3000.0 | 1575.0 | 42 |
| **8** | 3082.5 | 1597.5 | 21 |
| **9** | 3066.0 | 1564.5 | 8 |
| **10** | 3285.0 | 1650.0 | 21 |

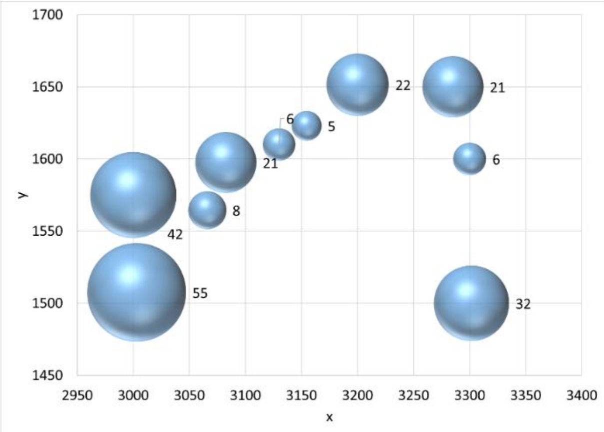


*Figure 5. Bubble chart representing the locations of pickup points and the amount of goods (number of pickup operations) to be collected*

KDE can be used to estimate the probability density function (PDF) of pickup point locations by smoothing discrete data points into a continuous distribution. Given a set of known pickup point coordinates and their weight factors representing the numbers of performed collection operations, KDE applies a Gaussian kernel function to each point, spreading its influence over a defined bandwidth. This results in a smooth density surface where higher values indicate regions with a greater concentration of pickup points. The choice of bandwidth is crucial, as a small bandwidth may capture too much local variation, while a large bandwidth can overly smooth the distribution. Fig. 6 shows the wireframe surface chart of the estimated probability density function of pickup points, while Fig. 7 demonstrates the 2-dimensional surface chart representing the estimated probability density function of pickup points in the 2-dimensional search space.

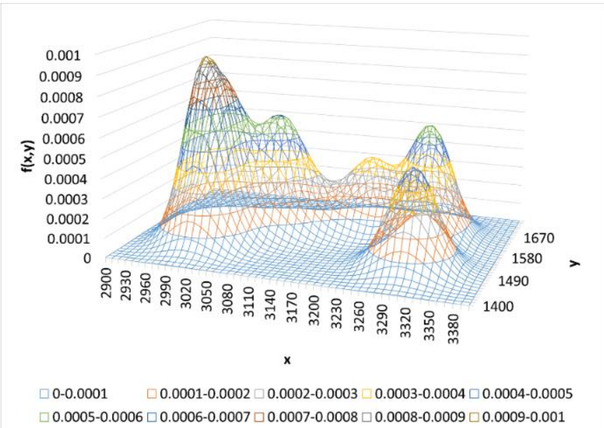


*Figure 6. Wireframe surface chart representing the estimated probability density function of pickup points in the 2-dimensional search space (in a map)*

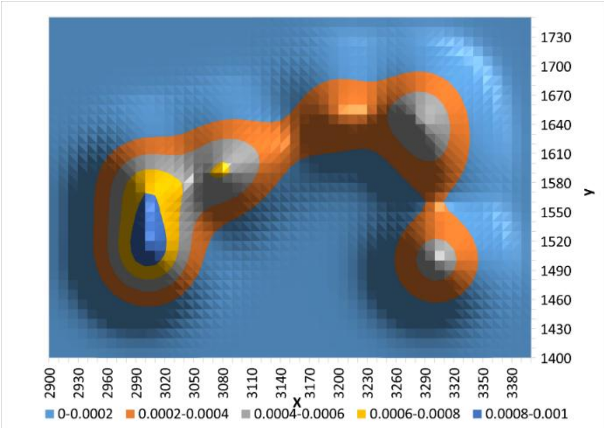


*Figure 7. 2-dimensional surface chart representing the estimated probability density function of pickup points in the 2-dimensional search space (in a map)*

The first phase of the optimization of the location of the collection center is to define the initial location of the collection center as the weighted centroid of pickup points:

$$x^{(0)} = \frac{\sum_{i=1}^{10} w_i \cdot x_i}{\sum_{i=1}^{10} w_i} = 3118.39\ m, \tag{18}$$

$$y^{(0)} = \frac{\sum_{i=1}^{10} w_i \cdot y_i}{\sum_{i=1}^{10} w_i} = 1566.44\ m. \tag{19}$$

Then, the update step at each iteration is given by:

$$x^{(k+1)} = \frac{\sum_{i=1}^{10} \frac{w_i \cdot x_i}{d_i^{(k)} = \sqrt{(x^{(k)} - x_i)^2 + (y^{(k)} - y_i)^2}.}}{\sum_{i=1}^{10} \frac{w_i}{d_i^{(k)} = \sqrt{(x^{(k)} - x_i)^2 + (y^{(k)} - y_i)^2}.}}, \tag{20}$$

$$y^{(k+1)} = \frac{\sum_{i=1}^{10} \frac{w_i \cdot y_i}{d_i^{(k)} = \sqrt{(x^{(k)} - x_i)^2 + (y^{(k)} - y_i)^2}.}}{\sum_{i=1}^{10} \frac{w_i}{d_i^{(k)} = \sqrt{(x^{(k)} - x_i)^2 + (y^{(k)} - y_i)^2}.}}, \tag{21}$$

This iterative process continues until the termination criterion is met, which is the following:

$$\varepsilon \leq 2\ m. \tag{22}$$

This termination criterion is fulfilled in the 6$^{th}$ iteration, therefore the optimal location for the collection center is the following:

$$x^{(6)} = 3077.60\ m, \tag{23}$$

$$y^{(6)} = 1572.31\ m. \tag{24}$$

Fig. 8 shows the initial position of the collections center, calculated as the weighted centroid of the pickup points and the optimal location resulted in the 6th iteration of the Weiszfeld algorithm.

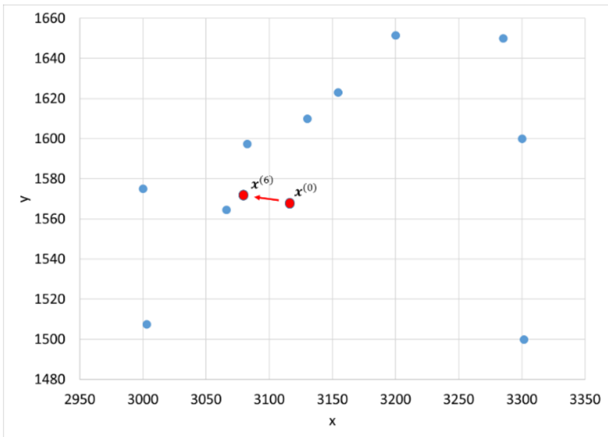


*Figure 8. Initial position of the collection center, calculated as the weighted centroid of the pickup points and the optimal location resulted in the 6th iteration of the Weiszfeld algorithm*

The next phase of the optimization is to apply the Weisfeld algorithm in the case of the heatmap, where the estimated probability density function of pickup points in the 2-dimensional search space is calculated. Based on Equations (8-14) we can calculate the optimal position of the collection center, which is as follows:

The termination criterion $\varepsilon \leq 2\ m$ is fulfilled in the 6th iteration, therefore the optimal location for the collection center is the following:

$$x^{(6)} = 3102{,}79\ m, \tag{25}$$

$$y^{(6)} = 1577.44\ m. \tag{26}$$

The convergence of the $\left\| x^{(k+1)} - x^{(k)} \right\|$ value is shown in Fig. 9.

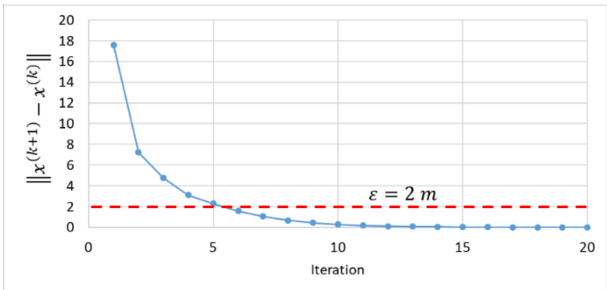


*Figure 9. The convergence of the difference between the results of two consecutive iteration steps.*

The initial position of the collection center, calculated as the weighted centroid of the estimated probability density function and the optimal location resulted in the 6th iteration of the Weiszfeld algorithm on the heatmap is shown in Fig. 10.

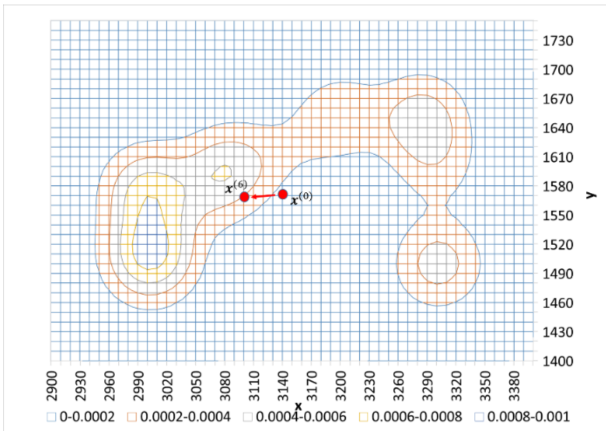


*Figure 10. Initial position of the collection center, calculated as the weighted centroid of the the estimated probability density function and the optimal location resulted in the 6th iteration of the Weiszfeld algorithm on the heatmap.*

The proposed heatmap-based approach effectively captured the spatial distribution of collection tasks, providing a more comprehensive representation than discrete point-based methods. The Weiszfeld algorithm, as the solution method of the Fermat-Weber problem successfully found the optimal collection center location by minimizing the total distances across the heatmap. Compared to conventional discrete optimization, the heatmap approach produced similar results but offered greater adaptability to varying demand distributions. The numerical experiment showed that the heatmap-based solution deviated by less than 5% from the optimal location computed using individual pickup points. The heatmap representation allowed for a smoother transition between facility locations when adjusting for new demand patterns. Unlike point-based methods, which can lead to abrupt shifts in facility placement, the heatmap approach ensured a gradual adaptation to demand fluctuations. Potential future research direction are the followings: (1) using real-time data streams from IoT sensors and GPS tracking could improve responsiveness to fluctuating demand patterns; (2) extending the method to multi-echelon logistics networks could provide insights into optimizing not only collection centers but also distribution hubs; (3) integrating machine learning techniques to predict future demand trends from historical data could make the heatmap approach even more dynamic.

## 5. ABBREVIATIONS

The following abbreviations are used in this manuscript:

- AHP: Analytical Hierarchy Process,
- CODAS: Combinative Distance-based Assessment,
- FANP: Fuzzy Analytic Network Process,
- FWP: Fermat-Weber Problem,
- Mixed Integer Linear Programming,
- MOMILP: Multi-Objective Mixed-Integer Linear Programming
- WA: Weiszfeld Algorithm.

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