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MATHEMATICAL MODEL OF MATRIX PRODUCTION

IBOLYA HARDAI[1] – BÉLA ILLÉS[2] – ÁGOTA BÁNYAI TÓTH[3]

Abstract: *The task of production logistics is to move workpieces within the plant between workstations, and to supply the workstations with the raw materials, components, etc. required for the given production task. This is no different in an AGV-served matrix production that complies with Industry 4.0 principles. In the case of larger workpieces, they are transported by a dedicated AGV throughout the production process. In the production space, several AGVs perform their tasks simultaneously, thus several material handling tasks occur simultaneously. This paper presents a mathematical model in detail, which allows for optimization according to various aspects. With the help of the model, we specify the objective functions and the restrictions on them.*

Keywords: *logistics, matrix production, layout, optimization, mathematical model*

1. INTRODUCTION

The factory of the future requires a reliable, easy-to-change, flexible production system. Modular manufacturing involves several changes in design, system, and processes [1], but it usually means that manufacturing must be divided into separate cells instead of a continuous line. Manufacturing and assembling the modules at separate stations allows for greater flexibility in the overall output, including changing product options or changing demand. The intelligent matrix production provides a solution to these.

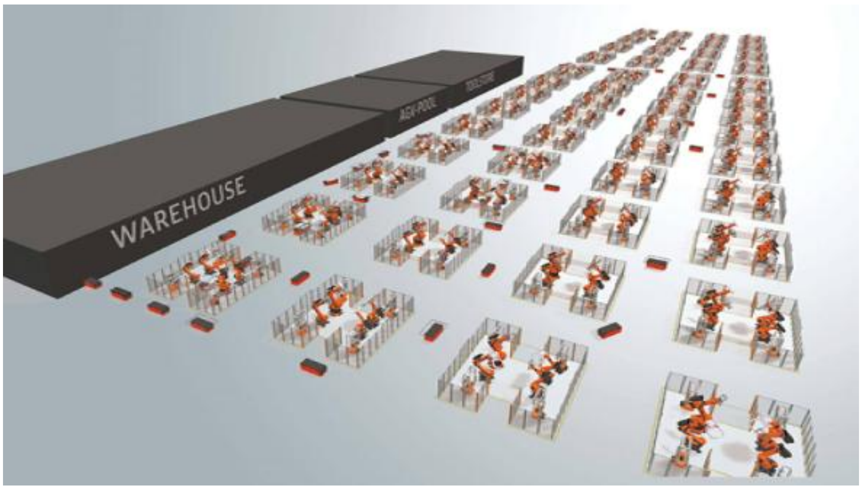

Figure 1. Matrix production [2]

[1]PhD student, University of Miskolc, Institute of Logistics, Hungary
hardaiibolya@gmail.com
[2] professor, University of Miskolc, Institute of Logistics, Hungary
bela.illes@uni-miskolc.hu
[3] professor, University of Miskolc, Institute of Logistics, Hungary
agota.banyaine@uni-miskolc.hu

The classic assembly line is not suitable for high economic efficiency in terms of future diversified production requirements. The goal is to maintain the smoothness of the production process while eliminating equal cycle times. Matrix production provides a sufficiently flexible transport system dynamics for the material supply to the work cells. The control system can cope with failures and thus the production system can be considered extremely robust [3].

The matrix production concept allows the production cells to be freely interconnected and thus removes the limitations of linear production. Many product variants can be produced in a single production structure. The duration of a single process step no longer determines the entire production cycle. Rather, process steps are executed independently of each other to create an infrastructure that is focused on processes and capacity requirements. However, the implementation of production cells requires control methods that allow for a capability-based description of the machining processes. This is the basis for rapid reconfiguration of the cell and efficient adaptation to new production tasks [4].

To achieve this, end-to-end data integration in the supply chain [5], automated material flow and real-time communication are needed. This requires the development and implementation of new manufacturing structures and solutions.

In a matrix production system, flexible, configurable manufacturing and assembly cells are arranged in a grid layout, and in-plant supply is based on autonomous vehicles. Adaptive and flexible material handling solutions are needed to meet the dynamically changing supply needs of standardised and categorised manufacturing cells [6].

Key features of matrix production:

- The modules for manufacturing are located at points on a grid,
- the same operation can be performed alternatively at several workstations,
- the material flow is completely flexible,
- service can be provided in several ways,
- the system can handle large quantities,
- the system has a sufficiently good resilience against fluctuations in production volume.

The main elements of a matrix production factory are:

- the production cells, the transport routes, the nodes, the entrance (ENTRANCE), the exit (EXIT), the warehouse (WAREHOUSE), the AGV warehouse (AGV-POOL), the tool warehouse (TOOLSTORE) [2],
- the workpieces, the devices, the tools, the components, the subassemblies, etc.,
- the control system.

The area suitable for carrying out the task(s) involved in a technological process is the production cell. The specific physical locations of the cells are determined considering the required cell size and transport path width. Within the cells, rotary tables are located for storing parts, tool holders and robots performing certain processes. These production cells can be individually extended with process-specific equipment. Many different processes can be integrated into a single cell, e.g. assembly, welding, gluing, cutting, soldering, riveting.

Since the number of technological processes may be less than the number of available production cells, it is possible for multiple cells to perform the same operation in parallel. In the event of an unexpected failure or planned shutdown, another cell can take over the tasks of the failed cell, ensuring uninterrupted production.

In reality, machines may have different installation characteristics and specific requirements, so not every layout unit may be suitable for placing any equipment. In the model we examined; to increase flexibility, we applied the following conditions:

- cells of identical size were created,
- the cells are equipped with basic robots so any technological operation for any product to be manufactured can be performed in any cell after the necessary retooling; in other words, the cells are standardized with product-neutral equipment and product-specific basic functions,
- the same type of operations can be performed in each cell at the same time for all products,
- if multiple operations, tasks, or technological steps are carried out in one cell, they are considered as a single unit (one operation, etc.) from the model's perspective.

The factory has one entrance where the workpieces arrive and one exit leading to the finished goods warehouse. Additionally, there are predefined locations for an AGV pool (AGV-POOL), a parts warehouse (WAREHOUSE), and a tool store (TOOLSTORE) (Fig. 1).

A dedicated AGV transports a given workpiece through the entire production process from the entrance to the exit.

3. General mathematical model

In the following subsections, we present the parameter sets and constraints of the optimization options according to various aspects, and the objective functions constructed using them. Two cells are considered different in the following if they perform different manufacturing operations.

3.1. Optimization of the layout of manufacturing cells

Target: To minimize all material handling work between cells, considering the distance between them and the material flow.

Target function:

$$Z_1 = \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{p=1}^{P}\sum_{q=1}^{P}(f_{i,j} \cdot d_{p,q} \cdot L_{i,p} \cdot L_{j,q}) \rightarrow \text{min.} \qquad p \neq q \tag{1}$$

where

$f_{i,j}$ – material flow intensity between the i-th and j-th cells,

p, q – two different positions,

$d_{p,q}$ – distance between the p-th and q-th positions,

$L_{i,p} \in \{0,1\}$ – binary variable that tells whether the i-th cell is placed in the p-th position,

$L_{j,q} \in \{0,1\}$ – binary variable that tells whether the j-th cell is placed in the q-th position,

M – number of unique production cells,

P – number of potential locations.

Constraints:

1. Cell-position assignment:

$$\sum_{p=1}^{P} L_{i,p} = 1 \qquad i = 1,2,\dots,M \in \mathbb{Z}^+ \tag{2}$$

Each cell must be assigned exactly one position.

2. Position-cell assignment:

$$\sum_{i=1}^{M} L_{i,p} \leq 1 \qquad p = 1,2,\dots,P \in \mathbb{Z}^+ \tag{3}$$

Each position can have a maximum of one cell assigned to it.

3.2. Optimization of the AGV routes

Target: Minimize the delivery time of AGVs based on route length and average speed, assuming that the layout of the production cells is already known.

Target function:

$$Z_2 = \max_x \sum_{x=1}^{N_x} \sum_{i=0}^{\beta_{x,y}+2} \frac{d_{x,y,i,i+1}}{\overline{v_{x,y,i,i+1}}} \rightarrow min \tag{4}$$

where

X – number of AGVs,

K – number of products to be produced,

$\boldsymbol{z} = [z_1; z_2; \dots; z_K]^T$ – sequence of products to be produced,

$R_{x,y,z_k} \in \{0,1\}$ – k-th product is the y-th load of the x-th AGV,

N_x – number of loads transported independently by the x-th AGV,

Θ – production cells, AGV-POOL, ENTRANCE, EXIT points,

$\beta_{x,y}$ – number of manufacturing operations for the y-th load of the x-th AGV,

$\boldsymbol{s}_{x,y} = \left[\Theta_{x,y,0}; \Theta_{x,y,1}; \Theta_{x,y,2}; \Theta_{x,y,3}; \dots; \Theta_{x,y,\beta_{x,y}+1}; \Theta_{\beta_{x,y}+2}; \Theta_{\beta_{x,y}+3}\right]^T$ – the points of the entire route of the x-th AGV for the y-th shipment in the order of their visit,

$d_{x,y,i,i+1}$ – the length of the route between the points $\Theta_{x,y,i}$ and $\Theta_{x,y,i+1}$,

$\overline{v_{x,y,i,i+1}}$ – the average speed of the AGV on the route between the points $\Theta_{x,y,\mathrm{i}}$ and $\Theta_{x,y,\mathrm{i}+1}$,

t_{0,z_k} – the time elapsed from the start of production to the start of production of the k-th product.

For each product, the AGV starts from the AGV-POOL ($\Theta_{x,y,0}$) and first goes to the entrance (ENTRANCE, $\Theta_{x,y,1}$), after loading the workpiece, it visits the $\beta_{x,y}$ cells required for the production of the product ($\Theta_{x,y,2}$ ... $\Theta_{\beta_{x,y}+1}$), then arrives at the EXIT point at $\Theta_{\beta_{x,y}+2}$ after unloading the finished product, it finally moves to the AGV-POOL at station $\Theta_{\beta_{x,y}+3}$. We assume that at least 3 operations are required to produce each product, therefore the value of $\beta_{x,y}$ can be a natural number between 3 and 8.

Constraints:

1. One product delivered by one AGV

$$\sum_{y=1}^{N_x}\sum_{x=1}^{X} R_{x,y,z_k} = 1 \qquad k = 1,2,\dots,K \in \mathbb{Z}^+ \tag{5}$$

2. One AGV delivers one product

$$\sum_{k=1}^{K} R_{x,y,z_k} = 1 \qquad x = 1,2,\dots,X \in \mathbb{Z}^+, y = 1,2,\dots,N_x \in \mathbb{Z}^+ \tag{6}$$

3. Production of products starts in the order of the production list.

$$\text{for all } i < j \text{ it is true that } t_{0,z_i} < t_{0,z_j} \qquad i,j = 1,2,\dots,K \in \mathbb{Z}^+ \tag{7}$$

3.3. AGV sequence optimisation

Target: Optimise the routing sequence of AGVs to minimise production lead times.

Target function:

$$Z_3 = \sum_{x=1}^{X} t_x(\Delta s,\tau) \rightarrow min \tag{8}$$

where

t_x – cumulative production times of workpieces delivered by the x-th AGV,
X – number of AGVs,
Δs – minimum following distance between two AGVs,
τ – the priority mode used.

It should be considered:

- the sequential departure of AGVs (priority),
- the possible waiting times,
- safe operation (following distance),
- transport times required to service production operations.

Constraints:

1. Priority management:
   Priority is high if the AGV is blocking a lot of people from moving forward.
   - no priority is applied ($\tau = P0$),
   - he AGV with a higher priority gets a route earlier ($\tau = P+$),
   - the AGV with a lower priority gets a route earlier ($\tau = P-$).
2. Avoid congestion, overlapping, collisions:
   The distance between two AGVs on each route section must be at least Δs. When arriving at a junction, the movement can proceed in the order according to the fixed rules (e.g.: the AGV that arrived earlier has priority, while the other one waits. If they arrive there at the same time, the AGV with a lower number can proceed first, but the order can also be determined according to the priorities).

3.4. Integrated optimisation

Target: Minimizing the total production time.

Target function:

$$t_{prod} = \max_{x}(t_x) \rightarrow min \quad (9)$$

where

t_x – total production times of the workpieces transported by the x-th AGV,

X – number of AGVs,

K – number of products to be produced,

$\boldsymbol{z} = [z_1; z_2; \dots; z_K]^T$ – sequence of products to be manufactured,

$R_{x,y,z_k} \in \{0,1\}$ – whether the y-th load of the x-th AGV is the k-th product

N_x – number of loads transported independently by the x-th AGV,

Θ – production cells, AGV-POOL, ENTRANCE, EXIT points,

β_{z_k} – number of manufacturing operations for product z_k,

$\boldsymbol{s}_{x,y} = \left[\Theta_{x,y,0}; \Theta_{x,y,1}; \Theta_{x,y,2}; \Theta_{x,y,3}; \dots; \Theta_{x,y,\beta_{z_k}+1}; \Theta_{\beta_{z_k}+2}; \Theta_{\beta_{z_k}+3}\right]^T$ – the points of the entire route of the x-th AGV for the y-th shipment in the order of their visit,

C – the number of route sections,

w_c – he c-th route section,

s_{w_c} – the length of the w_c route section,

Q – the number of section boundary points,

Φ_q – he q-th section boundary point,

$d_{x,y,i,i+1}$ – the length of the route section during the transportation of the y-th cargo of the x-th AGV between the points $\Theta_{x,y,i}$ and $\Theta_{x,y,i+1}$,

$\overline{v_{x,y,i,i+1}}$ – the average speed of the AGV during the transportation of the y-th cargo of the x-th AGV between the points $\Theta_{x,y,i}$ and $\Theta_{x,y,i+1}$,

$T_{x,y,i,i+1,w_c} \in \{0,1\}$ – whether the x-th AGV uses the road section w_c between points $\Theta_{x,y,i}$ and $\Theta_{x,y,i+1}$ during the transportation of its cargo y,

$W_{x,y,i,i+1,\Phi_q} \in \{0,1\}$ – whether the x-th AGV touches point Φ_q during the transportation of its cargo y between points $\Theta_{x,y,i}$ and $\Theta_{x,y,i+1}$,

t_{0,z_k} – the time elapsed from the start of production to the start of production of the k-th product.

Let's break down the road network that can be traversed by AGVs into sections as follows: at every point where the AGV reaches a junction and/or can change direction, as well as at the entrance, exit, AGV storage, tool storage and raw material storage, we create a section boundary point ($\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_Q$), and then assign a serial number to the road sections between the resulting points ($w_1, w_2, \dots, w_C$).

The number of route sections and section boundary points can be determined based on the factory's characteristics and the factory's "map".

During the production of a product, the AGV status indicators can be as follows:

- loading operation is taking place at the entrance,
- AGV is waiting for route permission (e.g. it wants to go from cell i to cell j),
- AGV is waiting for route (e.g. at an intersection),
- performing a material handling task,
- standing in a cell, production operation is in progress,
- loading operation is taking place at the exit,
- spends technical time in AGV-POOL.

According to the previous list, the following times can be distinguished if the x-th AGV is observed during the whole period of production:

- Loading times at the entrance:

$$t_x^{Rakfel} = \sum_{y=1}^{N_x} t_y^{Rakfel} \tag{10}$$

- Waiting times for route permission:

$$t_x^{Eng} = \sum_{y=1}^{N_x} \sum_{i=0}^{\beta_{z_k}+2} t_{\Theta_{x,y,i},y}^{Eng} \tag{11}$$

$t_{\Theta_{x,y,i},y}^{Eng}$ denotes the waiting time for route clearance for the y-th load of the x-th AGV at $\Theta_{x,y,i}$.

- Waiting times for a route:

$$t_x^{Útv} = \sum_{y=1}^{N_x} \sum_{i=0}^{\beta_{x,y}+2} \sum_{q=1}^{Q} (t_{\Theta_{x,y,i},\Theta_{x,y,i+1},\Phi_q,y}^{Útv} \cdot W_{x,y,i,i+1,\Phi_q}) \tag{12}$$

$t_{\Theta_{x,y,i},\Theta_{x,y,i+1},P_q,y}^{Útv}$ denotes the waiting time for the route for the x-th AGV and its y-th load between the points $\Theta_{x,y,i}$ and $\Theta_{x,y,i+1}$ at point Φ q.

- Material handling times:

$$t_x^{Száll} = \sum_{y=1}^{N_x} \sum_{i=0}^{\beta_{x,y}+2} \frac{\overline{\overline{d_{x,y,i,i+1}}}}{v_{x,y,i,i+1}} = \sum_{y=1}^{N_x} \sum_{i=0}^{\beta_{x,y}+2} \sum_{c=1}^{C} \frac{s_{w_c} \cdot \overline{T_{x,y,i,i+,1,w_c}}}{\overline{v_{x,y,i,i+1}}} \tag{13}$$

- Production times (in cells):

$$t_x^{Gyárt} = \sum_{y=1}^{N_x} \sum_{i=2}^{\beta_{x,y}+1} t_{i,y}^{Gyárt} \tag{14}$$

- Loading times at the exit:

$$t_x^{Rakle} = \sum_{y=1}^{N_x} t_y^{Rakle} \tag{15}$$

- Technical times in AGV-POOL:

$$t_x^{Techn} = \sum_{y=1}^{N_x} t_y^{Techn} \tag{16}$$

The x-th AGV's time in production:

$$t_x = t_x^{Rakfel} + t_x^{Eng} + t_x^{Útv} + t_x^{Száll} + t_x^{Gyárt} + t_x^{Rakle} + t_x^{Techn} \tag{17}$$

The target function:

$$t_{gyártás} = \max_x(t_x) \to min \tag{18}$$

$$t_{gyártás} = \max_x\Big(\sum_{y=1}^{N_x} t_y^{Rakfel} + \sum_{y=1}^{N_x}\sum_{i=0}^{\beta_{z_k}+2} t_{\Theta_{x,y,i},y}^{Eng} + \\ + \sum_{y=1}^{N_x}\sum_{i=0}^{\beta_{x,y}+2}\sum_{q=1}^{Q}(t_{\Theta_{x,y,i},\Theta_{x,y,i+1},\Phi_q,y}^{Útv} \cdot W_{x,y,i,i+1,\Phi_q}) + \\ + \sum_{y=1}^{N_x}\sum_{i=0}^{\beta_{x,y}+2}\sum_{c=1}^{C}\frac{s_{w_c} \cdot T_{x,y,i,i+,1,w_c}}{\overline{v_{x,y,i,i+1}}} + \sum_{y=1}^{N_x}\sum_{i=2}^{\beta_{x,y}+1} t_{i,y}^{Gyárt} + \sum_{y=1}^{N_x} t_y^{Rakle} + \\ + \sum_{y=1}^{N_x} t_y^{Techn}\Big) \quad \to min \tag{19}$$

Constraints:

1. Cell-position assignment:

$$\sum_{p=1}^{P} L_{i,p} \geq 1 \quad i = 1,2,\dots.,M \in \mathbb{Z}^+ \tag{20}$$

   Multiple positions can be assigned to a cell(operation).

2. Position-cell assignment:

$$\sum_{i=1}^{M} L_{i,p} \leq 1 \quad p = 1,2,\dots.,P \in \mathbb{Z}^+ \tag{21}$$

   Each position can have a maximum of one cell assigned to it.

3. One product delivered by one AGV:

$$\sum_{y=1}^{N_x}\sum_{x=1}^{X} R_{x,y,z_k} = 1 \qquad k = 1,2,\dots,\mathrm{K} \in \mathbb{Z}^+ \tag{22}$$

4. One AGV delivers one product:

$$\sum_{k=1}^{K} R_{x,y,z_k} = 1 \qquad \mathrm{x} = 1,2,\dots-,X \in \mathbb{Z}^+, \mathrm{y} = 1,2,\dots-,N_x \in \mathbb{Z}^+ \tag{23}$$

5. Production of the products starts in the order of the production list.

$$\text{for all i} < \text{j it is true that } t_{0,z_i} < t_{0,z_j} \qquad \text{i, j} = 1,2,\dots\text{-}, K \in \mathbb{Z}^+ \tag{24}$$

6. Any point Φ_q can only appear once on the AGV's path between two cells.

$$\sum_{y=1}^{N_x} \sum_{x=1}^{X} W_{x,y,i,i+1,\Phi_q} = 1$$
$$q = 1,2,\dots\text{-}, Q \in \mathbb{Z}^+, \mathrm{i} = 0,1,2,\dots\text{-}, \beta_{x,y} + 2 \in \mathbb{Z}^+ \tag{25}$$

7. Any w_c section can only appear once in the AGV's route between two cells.

$$\sum_{y=1}^{N_x} \sum_{i=0}^{\beta_{x,y}+2} T_{x,y,i,i+1,w_c} = 1$$
$$c = 1,2,\dots\text{-}, C \in \mathbb{Z}^+, i = 0,1,2,\dots\text{-}, \beta_{x,y} + 2 \in \mathbb{Z}^+ \tag{26}$$

8. Avoiding congestion (according to section 3.3)
9. An AGV enters a production cell at most once during the production process of a product.

$$\Theta_{x,y,i} \neq \Theta_{x,y,j}, ha\ i \neq j$$
$$i = 2,3,\dots\text{-}, \beta_{x,y} + 1 \in \mathbb{Z}^+, j = 2,3,\dots\text{-}, \beta_{x,y} + 1 \in \mathbb{Z}^+ \tag{27}$$

For any layout, the above $t_{gyártás}$ time can be specified. The layout where the production time is the lowest should be chosen.

4. Summary

In order to meet the dynamically changing customer needs, the development of industry 4.0 capabilities of manufacturing and service companies is essential to increase their efficiency and expand their capacity. To achieve this goal, it is necessary to enhance digitization, vertical and horizontal integration, and to operate new, efficient, flexible production systems that can satisfy individual customer needs with mass production.

Modern machine tools and machining centres have undergone significant development, but real progress towards high-efficiency individual production is the appearance of manufacturing systems as a cyber-physical system, associated with the intelligence and networking of machines and equipment. Matrix production offers such a state-of-the-art solution.

In this study, we have dealt with the mathematical modelling of matrix production and the related optimization possibilities. The model can be used to determine the target functions for material handling work, delivery time, and production time.

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