Advanced Logistic Systems – Theory and Practice, Vol. 19, No. 3 (2025), pp. 51-71.
https://doi.org/10.32971/als.2025.016

MODELLING AND SOLVING TRANSPORTATION PROBLEMS AS A DECISION SUPPORT TOOL IN LOGISTICS PROCESS DESIGN AND OPTIMIZATION

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Abstract: *This paper provides a comprehensive overview of classical and extended transportation problem-solving methods. It introduces well-known algorithms such as the North-West Corner Method, Least Cost Method, Vogel's Approximation Method, and the Simplex Method. The paper also presents real-world application areas, including home delivery of e-commerce packages, transportation of medical equipment and pharmaceuticals, transportation of fresh food products. Practical examples illustrate how these models function in real industrial contexts. Beyond the classical transportation model, the study explores its extensions by incorporating capacity and time constraints. These extended models are analyzed to understand their impact on cost and solution structure. The findings highlight the trade-offs between model complexity and operational feasibility in real-world transportation planning.*

Keywords: *transportation problem, North-West Corner Method, Least Cost Method, Vogel's Approximation Method, and the Simplex Method, optimization, linear solver.*

1. INTRODUCTION

Transportation problems constitute an essential part of the design tasks within material flow systems. Therefore, this chapter briefly reviews the main design tasks, following the framework outlined by Cselényi & Illés [1]. The design of material flow systems inherently involves the placement of factories, warehouses, and machines or storage units within a facility. When determining the location of such units, multiple criteria must be considered, and often more than one feasible solution may exist. Additionally, from an operational cost perspective, material handling cost and workload are critical factors. Therefore, among the possible facility layout and placement plans, those should be selected that meet the specified requirements and are optimal from the perspective of the material flow system.

The design models for the installation of machines and equipment can be categorized into two main groups: based on the shape of the objects to be placed, and whether the planning process is simple or integrated. The key difference between simple and integrated layout planning lies in their scope: in simple layout planning, only the assignment of objects to facility locations is carried out based on a given distance matrix; whereas in integrated planning, the material handling equipment is also selected, the transport paths are defined, and only then are the objects assigned to the appropriate facility locations.

Route planning involves determining optimal paths to ensure efficient material handling. Nowadays, the importance of forklift route planning is increasing, as significant cost savings can be achieved by designing optimal transport routes. A current trend is the retrofitting of traditional forklifts with automation capabilities, enabling either offline or online operation

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modes. This allows them to communicate with the computer systems controlling the logistics processes and the company's enterprise information system.

Driverless forklift systems (Automated Guided Vehicles, AGVs) can also be implemented, in which case route planning becomes particularly critical – especially when coordinating the operations of many AGVs.

The design of forklift-based material flow systems includes several elements: defining the boundaries of the material flow system, constructing the distance matrix, analyzing unit load formation, selecting the appropriate forklift type, designing loading/unloading stations, planning the forklift routes, determining the number of forklifts and unit loads required, and developing the forklift control system.

Route planning must take into account the intensity of material flow, the available fleet of forklifts, different types of transport routes, the forklifts' loading capacities, waiting times, and time constraints tied to specific operations.

Unit loads formed from individual items play a crucial role in facilitating the flow of materials and goods within logistics systems. Typically, a unit load (UL) refers to a grouping of materials or products placed on a unit load forming device (ULFD), which is then handled as a single entity throughout the logistics processes. Thus, the concept of a unit load inherently includes the product or material itself as well as the device that forms and supports the unit.

The functions and design of unit loads are always aligned with the logistical requirements of production and service processes. Unit loads can be classified according to various criteria, such as their content, function, or place of use. Based on their content, unit loads can be homogeneous, heterogeneous, or multi-level. According to their function, they may serve as RST (storage and transport) or technological unit loads. From the perspective of their application, they can be associated with procurement, distribution, production, or recycling processes.

The objective is to determine the number of devices required within a logistics system. To do this, it is important to consider that some material flow systems consist of machines operating in an intermittent or cyclical manner. Such machines result in discontinuous material flow. They operate in work cycles, meaning they can transport a specified quantity of material from the loading point to the unloading point during each cycle.

Before selecting an intermittently operating device, one must consider its mobility, as the equipment can either be mobile or stationary (fixed installation). The performance of intermittent transport systems depends on several factors: the weight of the load, the transport distance, and the intensity of material movement.

The reliability with which a material flow task can be carried out is one of the most critical evaluation criteria. Enhancing this reliability requires consideration of several factors, such as whether the material handling equipment operates in a continuous or intermittent mode. The degree and sophistication of mechanization also play a significant role, as human-related uncertainty cannot be disregarded.

To determine the reliability characteristics of system components, the calendar operating time must be used as the basis of analysis. In material handling systems, direct or indirect connections often exist between storage units and equipment, which can result in system-wide shutdowns if a single component fails.

There are various types of reliability models used to analyze such systems, including series models, parallel models, hybrid connection models, bridge connection models, and k-out-of-n (KooN) models [2].

The essence of the assignment problem lies in assigning m resources to n tasks in such a way that the defined objective function reaches its optimal value. In assignment problems, typical objective functions include cost minimization, efficiency maximization, or defect rate minimization [3]. It is essential that each resource is assigned to exactly one task, and each task is performed by exactly one resource. Furthermore, the assignment costs must be known in advance. Naturally, instead of cost, the objective function can be based on productivity indicators, defect ratios, or other quality metrics [4].

In practice, assignment problems are frequently used to allocate machines or operators to workpieces or tasks.

In contrast, the transportation problem aims to determine the optimal distribution of goods, where multiple production or supply sites must deliver to several demand points, such that the total transportation cost is minimized while satisfying supply and demand constraints. The problem assumes that each source has a defined supply, each destination has a specific demand, and the transportation costs between each source and destination are known. Practical applications include material movement between production, warehouses, and customers, or the optimization of raw material supply chains [5].

2. TRANSPORTATION AND ASSIGNMENT PROBLEMS

The core concept of the assignment problem is to allocate m resources to n tasks in such a way that a defined objective function reaches its optimum. In this context, the objective may be the minimization of cost, the maximization of efficiency, or the minimization of defect rates [3]. It is crucial that each resource is assigned to only one task, and each task is handled by only one resource. Additionally, the assignment costs must be known in advance. Naturally, instead of cost, the objective function can also be based on other metrics such as productivity indicators, defect rates, or other quality-related measures [4].

In practical applications, assignment problems are commonly used for matching machines or operators to workpieces or tasks. The goal of the transportation problem is to determine the optimal distribution of goods such that products are shipped from multiple production sites to multiple demand points while minimizing total transportation costs, and at the same time satisfying all supply and demand requirements. The problem assumes that each source has a specific supply, each destination has a defined demand, and the transportation costs between sources and destinations are known. This model can be applied, for example, in the movement of goods between production facilities, warehouses, and customers, or in the optimization of raw material supply chains [5].

2.1. Mathematical description of transportation problems

When formulating the optimization model of transportation problems, it is necessary to define the following key model elements [6]:

- input parameters,
- objective function(s),
- constraints, including any potential sign restrictions,
- decision variables.

In general, the input parameters of transportation problems include the distances or transportation costs between each pair of nodes (relations), the available supply at each source, and the demand at each sink (destination). Let y_{uv} denote the transportation distance or cost between supply point *u* and demand point *j*. Let *u=1…m* represent the supply nodes (sources) and *v=1…n* represent the demand nodes (sinks). Then, the objective function of the transportation problem – which aims to minimize the total transportation cost or total transportation distance – can be expressed as follows:

$$C = \sum_{u=1}^{m} \sum_{v=1}^{n} y_{uv} \cdot x_{uv} \rightarrow min., \tag{1}$$

where C denotes the objective function to be optimized, and x_{uv} represents the decision variable of the optimization problem, which satisfies the following conditions:

$$\forall\, u, v: x_{uv} \in \mathbb{R}. \tag{2}$$

In transportation problems, two key constraints must be defined. The first constraint relates to supply and ensures that no supply point can ship more than its available capacity. This condition can be formulated as follows:

$$\forall\, u: \sum_{v=1}^{n} x_{uv} \leq r_u, \tag{3}$$

where r_u is the amount of product available at supply point u (supply constraint).

The second constraint refers to demand and ensures that no demand point receives less than its required quantity; in other words, all demand must be fully satisfied:

$$\forall\, v: \sum_{u=1}^{m} x_{uv} \geq p_v, \tag{4}$$

where p_v is the demand at demand point v (demand constraint).

If the total supply equals the total demand, the problem is referred to as a balanced transportation problem. In this case, the general model defined by equations (1–4) can be expressed in the following form:

$$C = \sum_{u=1}^{m} \sum_{v=1}^{n} y_{uv} \cdot x_{uv} \rightarrow min., \tag{5}$$

$$\sum_{u=1}^{m} r_u = \sum_{v=1}^{n} p_v, \tag{6}$$

$$\forall\, u: \sum_{v=1}^{n} x_{uv} = r_u, \tag{7}$$

$$\forall\, v: \sum_{u=1}^{m} x_{uv} = p_v. \tag{8}$$

2.2. Balancing of transportation problems

If the transportation problem is unbalanced, two cases can be distinguished:
- in the first case, the total supply exceeds the total demand, while
- in the second case, the total demand exceeds the total supply.

2.2.1. Balancing a transportation problem when total supply exceeds total demand. In this case, the transportation problem can be balanced by introducing a virtual demand point whose demand is exactly equal to the surplus supply. If the transportation cost to this dummy demand point is set to zero from all supply points, the unbalanced transportation problem is transformed into a balanced one.

If the transportation problem is unbalanced and total supply exceeds total demand, it can be formulated with the following condition:

$$\sum_{u=1}^{m} r_u > \sum_{v=1}^{n} p_v. \tag{9}$$

In this case, the introduction of the virtual demand point can be defined in the following form:

$$p_{n+1} = \sum_{u=1}^{m} r_u - \sum_{v=1}^{n} p_v. \tag{10}$$

2.2.2. Balancing a transportation problem when total demand exceeds total supply. There is no feasible solution to this problem, as it is not possible to satisfy all demand due to limited supply. However, in order to make the transportation problem solvable (even if not all demand is fulfilled), a virtual supply point can be introduced. This point does not exist in reality and does not provide actual goods, but it allows the transportation problem to be balanced. In such cases, unmet demand may incur a penalty cost.

If the transportation problem is unbalanced and total demand exceeds total supply, this condition can be formulated as follows:

$$\sum_{u=1}^{m} r_u < \sum_{v=1}^{n} p_v. \tag{11}$$

The introduction of the virtual supply point can then be defined in the following form:

$$r_{m+1} = \sum_{v=1}^{n} p_v - \sum_{u=1}^{m} r_u. \tag{12}$$

2.3. Application areas of transportation problems

Transportation problems have numerous application areas, some of which are discussed in this chapter.

2.3.1. Home delivery of e-commerce packages. Courier and postal services have long-established delivery systems that have evolved into a standalone industry. Initially, online shops delivered ordered packages to customers through the national postal service, typically with a turnaround of several days. Today, specialized logistics companies like GLS, DPD, or ExpressOne enable much faster delivery – often within 1-2 days – complemented by parcel locker or pickup point options.

From a mathematical standpoint, the operation of courier services can be interpreted as solving classic transportation problems. In such scenarios, packages must be transported from various warehouses or distribution centers to customers or pickup points in a way that:

- accounts for the available supply (capacity) at the source points (warehouses),
- satisfies the demand at the destinations (customers or pickup points),
- considers transportation costs associated with each delivery route (e.g., distance, time, fees),
- minimizes the total transportation cost [7].

For international deliveries, the transportation task becomes multi-stage: packages pass through distribution hubs in several countries, which can be modeled as a sequence of interdependent transportation problems.

Additional complexities for logistics providers include:

- Time windows: Deliveries must occur within specific time intervals, turning the problem into a time-constrained transportation task.
- Returns management: Undelivered packages need to be sent back, generating a reverse transportation problem, where destinations and sources switch roles.
- Multiple delivery attempts: Failed deliveries trigger repeated attempts, adding extra transportation costs that must be optimized at the system level.

In summary, modern courier and parcel delivery systems are built on dynamic and complex variations of classical transportation optimization problems. These systems must consider not only cost minimization but also temporal and capacity constraints [8, 9].

2.3.2. Transportation of Medical Equipment and Pharmaceuticals. The transportation of medical devices and pharmaceuticals is governed by strict regulations to ensure the safety, quality, and efficacy of the products. Throughout the logistics process, it is essential to maintain continuous tracking, perform regular checks, and keep accurate documentation. This is particularly critical for temperature-sensitive or specially-handled medications, where maintaining appropriate environmental conditions - such as a consistent cold chain - is vital.

From a mathematical perspective, these transportation processes can be modeled using the framework of classical transportation problems [10]. In such cases, the objective is to transport medical equipment or pharmaceuticals:

- from various production sites or warehouses (sources),
- to hospitals, pharmacies, or healthcare centers (destinations),

in a way that:

- respects the available stock at each source (supply),
- fulfills the demand at each destination, and
- considers the transportation costs on each route (which may include additional fees for refrigerated or special handling),

while minimizing total transportation cost and fully satisfying all demands [11].

Special attention must be given to the following factors:

- Cold chain logistics requirements: Some destinations only accept deliveries via refrigerated transport. These routes can only be served using specialized vehicles, adding constraints and potential cost increases to the transportation model.
- Special handling requirements: Certain medicines require expedited delivery or specific handling conditions, which must be factored into the cost function as penalties or additional costs.
- Dual cost structures: Standard and special shipments often have different pricing schemes. This must be accounted for when developing an optimal transportation plan.

In summary, the distribution of medical products constitutes a complex transportation problem where strict adherence to handling conditions is just as critical as cost minimization during the optimization process [12].

2.3.3. Transportation of Fresh Food Products. The planning of fresh food transportation depends on several factors, such as which products are in short supply at specific retail locations, and from which sources these products can be procured and at what distance. In mathematical terms, the distribution of food products can be modeled as a classic transportation problem [13]. The objective is to:

- transport food products from various production sites, warehouses, or wholesale distribution centers (sources),
- to retail stores or final points of sale (destinations),

in a way that takes into account the available quantities at the sources (supply), satisfies the demand at the stores, and minimizes the total transportation cost associated with each route (including distance, transit time, and any special transport requirements) [14].

This problem becomes more complex due to the following considerations:

- Special transport requirements: Refrigerated or frozen products must be transported using a cold chain, which limits the number of viable transport routes and increases individual shipment costs.
- Time-critical deliveries: For fresh products, strict delivery time constraints must be respected, turning the problem into a time-window-based transportation task.
- Packaging characteristics: The type and extent of packaging required for certain goods can influence transportation costs, which should also be integrated into the cost function.

In summary, the design of food supply chain logistics represents a specialized and complex variant of the classical transportation problem, where, in addition to cost minimization, strict quality, environmental, and time-related constraints must be satisfied [15].

2.3.4. Transportation of vehicle components. As private individuals, we typically order parts from distributor companies; however, these components originally travel from manufacturing sites to the distributors. The logistics systems of the automotive industry often operate based on lean and just-in-time (JIT) principles, meaning components are shipped only when they are actually needed. This minimizes warehousing costs and reduces the need for inventory.

Proper packaging is also a key focus during the transportation of vehicle components to protect sensitive parts from damage. Depending on urgency, transportation may occur by road, rail, or air. The return of defective components and all transport-related activities must be thoroughly documented to ensure full traceability [16].

From a mathematical perspective, the movement of components can be modeled as a classic transportation problem. The objective in this case is to deliver vehicle parts from manufacturing sites (sources) to distributors, repair shops, or end-users (destinations), while:

- respecting the available inventory at the sources (supply),
- satisfying the demand at the destinations (order quantities), and
- minimizing the total transportation cost associated with each route (which may vary depending on the transportation mode: road, rail, or air).

Additional complicating factors include:

- Differences in transportation modes: Depending on urgency, different modes must be chosen, each with distinct costs and delivery times (e.g., air freight is faster but more expensive).
- Return logistics: The return of defective parts must be incorporated into the logistics system, generating additional transport tasks and associated costs.
- Packaging requirements: Certain components require special packaging, which can increase transportation costs and should be reflected in the cost function.

In summary, the supply of vehicle components represents a complex transportation problem, where cost optimization must be achieved alongside speed, accuracy, and secure delivery requirements [17].

2.3.5. Waste transportation. Each landfill site typically serves multiple municipalities, and the transport of residential waste from these areas requires careful logistical planning. As a first step in the waste management process, the volume of waste generated by each municipality is assessed. Based on this data, transportation routes and schedules are optimized. Route planning takes into account the layout of each settlement and local traffic conditions to develop the most cost-effective solutions. During transportation, strict adherence to environmental and public health regulations is essential—such as ensuring that waste is moved in sealed containers.

From a mathematical perspective, the transport of municipal waste to landfill sites can be modeled using the classical transportation problem framework [18]. The objective is to:

- transport the waste generated in municipalities (sources),
- to the designated landfill sites (destinations),
- in such a way that the total transportation cost is minimized, considering:
  - the amount of waste accumulated in each municipality (supply),
  - the capacity of each landfill (demand),
  - and the transportation costs between each municipality-landfill pair (which may depend on distance, road type, and the characteristics of the required transport vehicles).

Additional factors influencing the problem include:

- Capacity constraints: Landfills have limited capacities, which must be strictly accounted for in the transportation plan.

- Environmental regulations: For instance, waste must be transported in sealed containers, necessitating specific vehicle types and resulting in higher transportation costs.
- Route-specific factors: The geographical layout and traffic conditions of municipalities—such as mountainous terrain or congested urban zones—can significantly affect transportation costs.

In summary, waste transportation planning represents a complex version of the classical transportation problem, where in addition to cost optimization, compliance with legal and environmental standards is a fundamental requirement [19].

2.4. Solution methods for transportation problems

The transportation problem is a type of linear programming problem that focuses on finding the most efficient way to distribute goods from multiple sources to multiple destinations while minimizing total transportation cost. Several methods have been developed to find initial feasible and optimal solutions to this problem. Below is a summary of the most commonly used methods, each with a brief description and corresponding process flow [6].

2.4.1. North-West Corner Method. The North-West Corner Method is a basic technique to find an initial feasible solution. It starts at the top-left (north-west) cell of the transportation table. Units are allocated to this cell as much as possible without exceeding supply or demand. Once supply or demand is satisfied, the method moves either right or down depending on which constraint is exhausted. This continues until all supply and demand constraints are fulfilled. It does not consider transportation costs, so the result is usually not optimal (see Fig. 1).

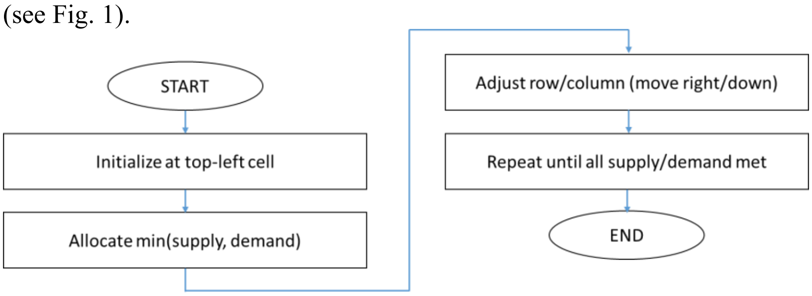

Figure 1. Flowchart of North-West Corner Method (Source: own edition)

2.4.2. Least Cost Method. The Least Cost Method improves upon the North-West Corner by considering transportation costs. It selects the cell with the lowest cost for allocation at each step. Then, it assigns as much as possible to this cell without violating supply or demand. Once either the row or column is fulfilled, it is crossed out and the process continues. Ties are broken arbitrarily or by choosing the next least cost. This yields a better initial solution than the North-West Corner method but still might not be optimal (see Fig. 2).

2.4.3. Vogel's Approximation Method. Vogel's Approximation Method is a heuristic that generally provides a very good initial feasible solution. For each row and column, it calculates a penalty — the difference between the two lowest costs. The row or column with the highest penalty is selected to make an allocation. Within that row/column, allocation is made to the cell with the lowest cost. Supply and demand are updated, and the penalties are recalculated. This continues until all allocations are completed (see Fig. 3).

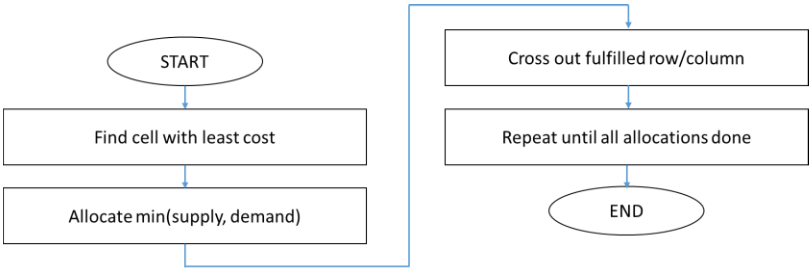

Figure 2. Flowchart of Least Cost Method (Source: own edition)

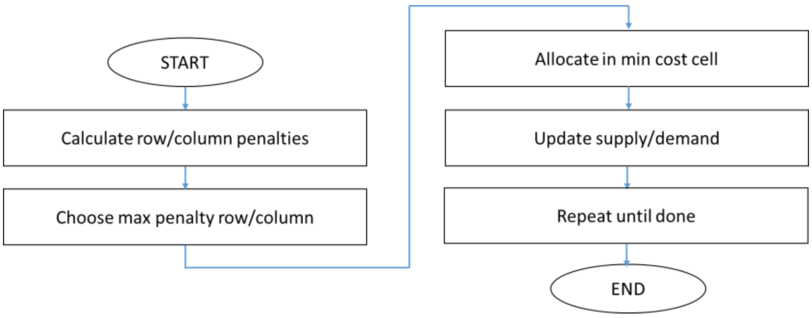

Figure 3. Flowchart of Vogel's Approximation Method (Source: own edition)

2.4.4. Simplex Method. The Simplex Method is used for optimizing linear programming problems, including transportation problems. After obtaining an initial basic feasible solution, Simplex iteratively improves the solution. It identifies entering and leaving variables based on optimality conditions. Pivoting is performed to move to a better corner point in the feasible region. The process continues until there is no further improvement in the objective function. It guarantees an optimal solution if one exists (see Fig. 4).

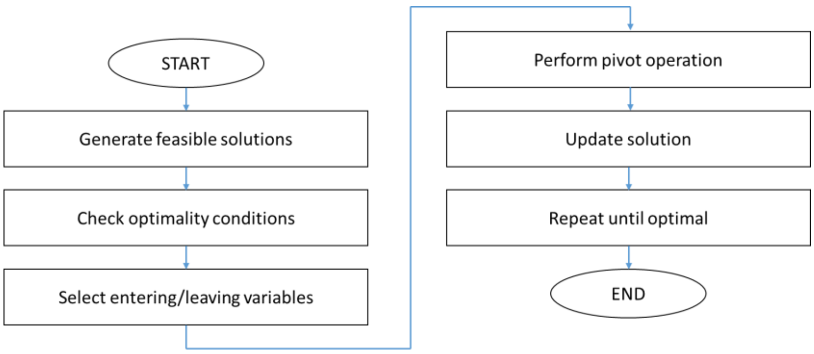

Figure 4. Flowchart of Simplex Method (Source: own edition)

3. OPTIMIZATION OF COMPLEX TRANSPORTATION PROBLEMS USING A LINEAR SOLVER

As demonstrated in the previous chapters discussing operations research methods, traditional transportation problems can be solved using the techniques introduced. However, when complex constraints are added – making the optimization problem more intricate and potentially non-linear – analytical methods may no longer be sufficient, and heuristic approaches might be necessary.

In this chapter, we examine which constraints increase the complexity of the model while still keeping the problem linear, allowing it to be solved using Excel's built-in Solver. For each example, we will present the mathematical model, followed by its representation in the Solver, along with the corresponding Excel worksheets.

To ensure the worksheets remain readable, we begin with a medium-sized (8×8) base problem to demonstrate how transportation problems of varying complexity can be solved using the evolutionary solver. All examples will involve balanced transportation problems, since transforming an unbalanced problem into a balanced one can be done as previously explained and does not increase the complexity of the task.

3.1. Solving the classical transportation problem

3.1.1. Mathematical model of the classical transportation problem. The objective function for an 8×8 transportation problem can be written in the following form:

$$C = \sum_{u=1}^{8}\sum_{v=1}^{8} y_{uv} \cdot x_{uv} \rightarrow min. \tag{13}$$

Since the transportation problem is balanced, the following condition can be taken into consideartion:

$$\sum_{u=1}^{8} r_u = \sum_{v=1}^{8} p_v. \tag{14}$$

The first constraint is the supply condition, which states that the amount shipped from each supply point must equal the amount available:

$$\forall\, u: \sum_{v=1}^{8} x_{uv} = r_u. \tag{15}$$

The second constraint is the demand condition, which states that the amount delivered to each demand point must equal the demand, meaning that all demands must be fully satisfied:

$$\forall\, v: \sum_{u=1}^{8} x_{uv} = p_v. \tag{16}$$

3.1.2. Optimization of a traditional transportation problem using a linear solver. The traditional transportation problem is a linear programming problem, and therefore it can be solved using a linear solver. The first step of the solution involves entering the known parameters of the problem, the objective function, the constraints, and the decision variables into an Excel worksheet:

- defining the identifiers of sources and sinks (C2:J2 and B3:B10),
- entering the unit transportation costs (C3:J10),
- specifying the available quantity of material at each source (K3:K10),
- entering the demand at each sink (C11:J11),
- selecting the cells for the decision variables (N3:U10),
- calculating the objective function, which is the sum of the products of corresponding elements from the unit transportation cost matrix and the decision variable matrix (K14),
- to specify the first constraint, calculating the row sums of the decision variable matrix (V3:V10),
- to specify the second constraint, calculating the column sums of the decision variable matrix (N11:U11).

The Excel worksheet described above is illustrated in Fig. 5.

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	58	19	44	54	20	57	22	58	332
F2	44	15	16	23	35	52	48	35	268
F3	28	13	46	50	17	40	14	14	222
F4	30	24	46	57	57	16	20	10	260
F5	28	34	51	21	20	40	39	24	257
F6	23	17	59	17	22	43	24	38	243
F7	43	33	51	29	30	46	26	50	308
F8	42	49	52	50	56	16	24	11	300
SUM	296	204	365	301	257	310	217	240	

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	0	0	75	0	257	0	0	0	332
F2	0	0	268	0	0	0	0	0	268
F3	0	204	0	0	0	0	18	0	222
F4	10	0	0	0	0	10	0	240	260
F5	43	0	0	214	0	0	0	0	257
F6	243	0	0	0	0	0	0	0	243
F7	0	0	22	87	0	0	199	0	308
F8	0	0	0	0	0	300	0	0	300
SUM	296	204	365	301	257	310	217	240	

Objective function: 43398

Figure 5. Excel worksheet for solving a traditional transportation problem using a linear solver (Source: own edition)

As illustrated in Fig. 5, the optimal solution obtained by the linear solver results in a total cost of 43,398 EUR.

The solution can be visually represented using a bipartite graph. For this purpose, we created and executed the following Python code on the online platform https://colab.research.google.com/:

```
import matplotlib.pyplot as plt
import networkx as nx
import numpy as np
# Optimal solution resulted by the Solver
matrix = np.array([
    [0, 0, 75, 0, 257, 0, 0, 0],
    [0, 0, 268, 0, 0, 0, 0, 0],
    [0, 204, 0, 0, 0, 0, 18, 0],
    [10, 0, 0, 0, 0, 10, 0, 240],
    [43, 0, 0, 214, 0, 0, 0, 0],
    [243, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 22, 87, 0, 0, 199, 0],
    [0, 0, 0, 0, 0, 300, 0, 0]
```

```
])
rows, cols = matrix.shape
# Create graph
G = nx.Graph()
# Add nodes
for i in range(rows):
    G.add_node(f"F{i+1}", bipartite=0)  # left-side nodes F1, F2, ..., F8
for j in range(cols):
    G.add_node(f"N{j+1}", bipartite=1)  # right-side nodes N1, N2, ..., N8
# Add edges based on matrix values
for i in range(rows):
    for j in range(cols):
        if matrix[i, j] > 0:
            G.add_edge(f"F{i+1}", f"N{j+1}", weight=matrix[i, j])
# Define positions
pos = {}
for i in range(rows):
    pos[f"F{i+1}"] = (0, -i)  # left-side nodes
for j in range(cols):
    pos[f"N{j+1}"] = (2, -j)  # right-side nodes
# Labels of edge labels
edge_labels = {(u, v): d["weight"] for u, v, d in G.edges(data=True)}
# Show graph
plt.figure(figsize=(8, 6))
nx.draw(
    G, pos,
    with_labels=True,
    node_size=600,
    node_color="lightblue",
    font_size=10,
    edge_color="gray"
)
nx.draw_networkx_edge_labels(
            G,     pos,     edge_labels=edge_labels,     font_size=8,     label_pos=0.2,
verticalalignment='center'
)
plt.tight_layout()
plt.show()
```

Based on the above code, Fig. 6 presents the optimal solution of the traditional transportation problem in the form of a bipartite graph.

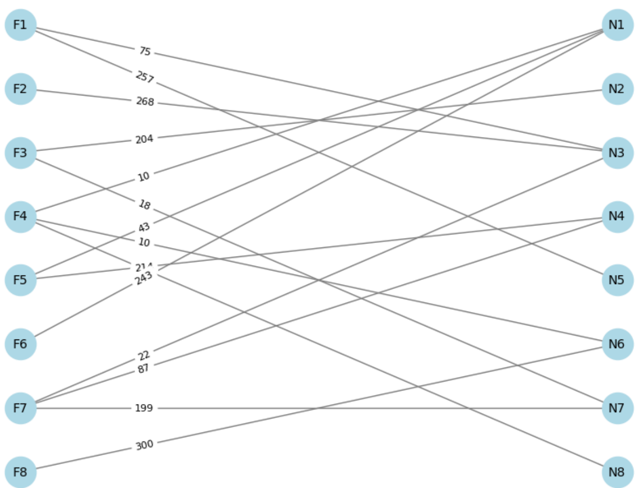

Figure 6. Optimal solution of the traditional transportation problem as a bipartite graph (Source: own edition)

3.2. Solving a capacity-constrained transportation problem

3.2.1. Mathematical model of the capacity-constrained transportation problem. The traditional model is extended with the following constraint, which defines that only a limited amount can be transported from each source to each sink. This can also be interpreted as a generalization of the capacity limitations of transport vehicles:

$$\forall\, u: x_{uv} \leq \alpha_{uv}^{max}, \tag{17}$$

where α_{uv}^{max} defines the quantity that can be transported from source u to sink v.

3.2.2. Optimization of a capacity-constrained transportation problem using a linear solver. The capacity-constrained transportation problem is still a linear programming problem, and therefore it can be solved using a linear solver. The first step of the solution involves entering the known parameters, the objective function, the constraints, and the decision variables into an Excel worksheet, as shown in Fig. 7.

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	58	19	44	54	20	57	22	58	332
F2	44	15	16	23	35	52	48	35	268
F3	28	13	46	50	17	40	14	14	222
F4	30	24	46	57	57	16	20	10	260
F5	28	34	51	21	20	40	39	24	257
F6	23	17	59	17	22	43	24	38	243
F7	43	33	51	29	30	46	26	50	308
F8	42	49	52	50	56	16	24	11	300
SUM	296	204	365	301	257	310	217	240	

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	0	36	30	0	100	0	166	0	332
F2	0	168	100	0	0	0	0	0	268
F3	206	0	0	0	0	0	16	0	222
F4	10	0	0	0	0	190	0	60	260
F5	30	0	0	70	157	0	0	0	257
F6	50	0	0	193	0	0	0	0	243
F7	0	0	235	38	0	0	35	0	308
F8	0	0	0	0	0	120	0	180	300
SUM	296	204	365	301	257	310	217	240	

	N1	N2	N3	N4	N5	N6	N7	N8
F1	332	332	30	332	100	332	332	332
F2	268	268	100	268	268	268	268	268
F3	222	100	222	222	222	222	222	222
F4	260	260	260	260	260	260	260	120
F5	30	257	257	257	257	257	257	257
F6	50	243	243	243	243	243	243	243
F7	308	308	308	308	308	308	80	308
F8	300	300	300	300	300	120	300	300

Objective function: 49486

Figure 7. Excel worksheet for solving a capacity-constrained transportation problem using a linear solver (Source: own edition)

The components of Fig. 7 contain the following data:

- definition of source and sink identifiers (C2:J2 and B3:B10),
- input of unit transportation costs (C3:J10),
- available quantity of material at each source (K3:K10),
- demand at each sink (C11:J11),
- quantity limits on shipments from sources to sinks (C14:J21),
- selection of cells for decision variables (N3:U10),
- calculation of the objective function, which is the sum of the products of the unit transportation cost matrix and the decision variable matrix (V14),
- for defining the first constraint, calculating the row sums of the decision variable matrix (V3:V10),

- for defining the second constraint, calculating the column sums of the decision variable matrix (N11:U11).

As illustrated in Fig. 6, the optimal solution obtained by the linear solver results in a total cost of 49,486 EUR.

The optimal solution of the capacity-constrained transportation problem is visualized as a bipartite graph in Fig. 8.

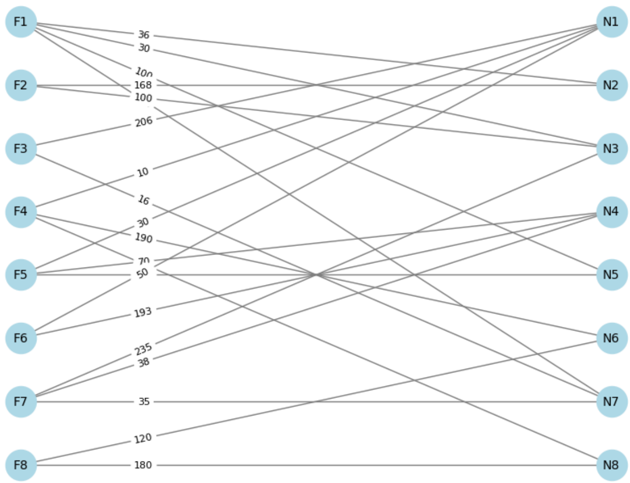

Figure 8. Optimal solution of the capacity-constrained transportation problem as a bipartite graph (Source: own edition)

3.3. Solving a time-constrained transportation problem

3.3.1. Mathematical model of the time-constrained transportation problem. The basic model is extended with an additional constraint. This defines that each source has a specified availability time, and each sink has a known time window during which the demand must be fulfilled. Therefore, a source can only deliver to a sink if the availability and demand time windows overlap. This corresponds to the time interval intersection problem, as the feasible transportation time window can be determined as follows:

$$\forall\, u, v: t^*_{uv} \in [max(\tau_u^{Bmin}, \tau_v^{Fmin}), min(\tau_u^{Bmax}, \tau_v^{Fmax})], \tag{18}$$

where t^*_{uv} defines the time interval within which it is possible to transport goods from source u to sink v., τ_u^{Bmin} denotes the start time of availability for source u, τ_u^{Bmax} denotes the final time of availability for source u, τ_v^{Fmin} denotes the start time of the receiving capacity of sink v and τ_v^{Fmax} denotes the final time of the receiving capacity of sink v.

This also means that source u can deliver to sink v only if the following condition is satisfied:

$$max(\tau_u^{Bmin}, \tau_v^{Fmin}) < min(\tau_u^{Bmax}, \tau_v^{Fmax}). \tag{19}$$

3.3.2. Optimization of a time-constrained transportation problem using a linear solver. The time-constrained transportation problem is still a linear programming problem and can therefore be solved using a linear solver. The first step of the solution involves entering the known parameters, the objective function, the constraints, and the decision variables into an Excel worksheet, as shown in Fig. 9.

The components of Fig. 9 contain the following data:

- definition of source and sink identifiers (C2:J2 and B3:B10),
- input of unit transportation costs (C3:J10),
- available quantity of material at each source (K3:K10),
- demand at each sink (C11:J11),
- lower bound of the sources' availability time window, expressed in calendar days (C13:J13),
- upper bound of the sources' availability time window, expressed in calendar days (C14:J14),
- lower bound of the sinks' acceptable delivery time window, expressed in calendar days (K16:K23),
- upper bound of the sinks' acceptable delivery time window, expressed in calendar days (L16:L23),
- selection of cells for decision variables (O3:V10),
- calculation of the objective function, which is the sum of the products of the unit transportation cost matrix and the decision variable matrix (W12),
- to define the first constraint, calculating the row sums of the decision variable matrix (W16:W23),
- to define the second constraint, calculating the column sums of the decision variable matrix (O24:V24),
- for the third constraint, creating a binary matrix that, identifies which source-sink combinations allow transportation.

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	58	19	44	54	20	57	22	58	332
F2	44	15	16	23	35	52	48	35	268
F3	28	13	46	50	17	40	14	14	222
F4	30	24	46	57	57	16	20	10	260
F5	28	34	51	21	20	40	39	24	257
F6	23	17	59	17	22	43	24	38	243
F7	43	33	51	29	30	46	26	50	308
F8	42	49	52	50	56	16	24	11	300
SUM	296	204	365	301	257	310	217	240	

min	7	15	10	20	7	8	5	10
max	15	17	14	26	25	27	19	20

	N1	N2	N3	N4	N5	N6	N7	N8	min	max
F1	1	1	1	0	1	1	1	1	5	18
F2	1	0	1	0	1	1	1	1	4	12
F3	1	1	1	0	1	1	1	1	12	16
F4	0	1	0	1	1	1	1	1	15	28
F5	0	0	0	1	1	1	0	0	22	29
F6	1	1	1	0	1	1	1	1	6	20
F7	1	0	1	0	1	1	1	1	2	12
F8	1	1	1	1	1	1	1	1	6	22

	N1	N2	N3	N4	N5	N6	N7	N8
F1	0	69	6	0	257	0	0	0
F2	0	0	268	0	0	0	0	0
F3	53	135	0	0	0	0	0	34
F4	0	0	0	0	0	54	0	206
F5	0	0	0	257	0	0	0	0
F6	243	0	0	0	0	0	0	0
F7	0	0	91	0	0	0	217	0
F8	0	0	0	44	0	256	0	0

Objective function: 45207

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	0	69	6	0	257	0	0	0	332
F2	0	0	268	0	0	0	0	0	268
F3	53	135	0	0	0	0	0	34	222
F4	0	0	0	0	0	54	0	206	260
F5	0	0	0	257	0	0	0	0	257
F6	243	0	0	0	0	0	0	0	243
F7	0	0	91	0	0	0	217	0	308
F8	0	0	0	44	0	256	0	0	300
SUM	296	204	365	301	257	310	217	240	

Figure 9. Excel worksheet for solving a time-constrained transportation problem using a linear solver (Source: own edition)

The optimal solution of the time-window-constrained transportation problem is visualized as a bipartite graph in Fig. 10.

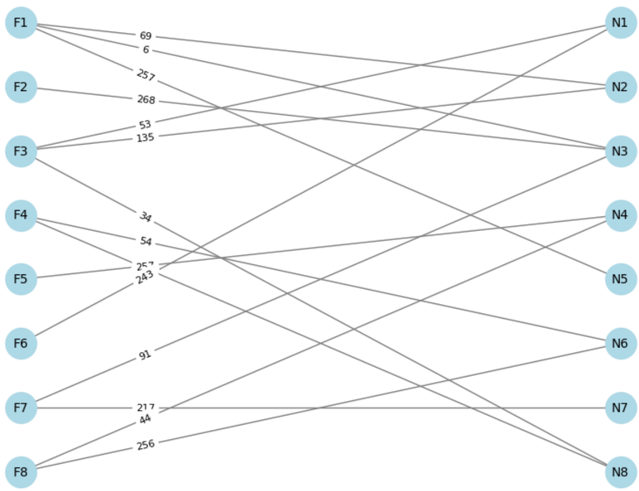

Figure 10. Optimal solution of the time-constrained transportation problem as a bipartite graph (Source: own edition)

3.4. Solving a time- and capacity-constrained transportation problem

3.4.1. Mathematical model of the time- and capacity-constrained transportation problem. Based on the previously outlined models, the general model of the time- and capacity-constrained transportation problem can be formulated as follows:

$$C = \sum_{u=1}^{8}\sum_{v=1}^{8} y_{uv} \cdot x_{uv} \rightarrow min. \tag{20}$$

$$\sum_{u=1}^{8} r_u = \sum_{v=1}^{8} p_v. \tag{21}$$

$$\forall\, u\colon \sum_{v=1}^{8} x_{uv} = r_u. \tag{22}$$

$$\forall\, v\colon \sum_{u=1}^{8} x_{uv} = p_v. \tag{23}$$

$$\forall\, u\colon x_{uv} \leq \alpha_{uv}^{max}, \tag{24}$$

$$\forall\, u,v\colon t_{uv}^{*} \in [max(\tau_u^{Bmin}, \tau_v^{Fmin}), min(\tau_u^{Bmax}, \tau_v^{Fmax})], \tag{25}$$

3.4.2. Optimization of a time- and capacity-constrained transportation problem using a linear solver. The time- and capacity-constrained transportation problem is still a linear programming problem, and therefore it can be solved using a linear solver. The first step of the solution involves entering the known parameters, the objective function, the constraints, and the decision variables into an Excel worksheet, as shown in Fig. 11.

The components of Fig. 11 contain the following data:

- definition of source and sink identifiers (C2:J2 and B3:B10),
- input of unit transportation costs (C3:J10),
- available quantity of material at each source (K3:K10),
- demand at each sink (C11:J11),
- capacity limits for quantities transported from sources to sinks (Y3:AF10),
- lower bound of the sources' availability time window, expressed in calendar days (C13:J13),
- upper bound of the sources' availability time window, expressed in calendar days (C14:J14),
- lower bound of the sinks' acceptable delivery time window, expressed in calendar days (K16:K23),
- upper bound of the sinks' acceptable delivery time window, expressed in calendar days (L16:L23),
- selection of cells for decision variables (O3:V10),
- calculation of the objective function, which is the sum of the products of the unit transportation cost matrix and the decision variable matrix (W12),
- to define the first constraint, calculating the row sums of the decision variable matrix (W16:W23),
- to define the second constraint, calculating the column sums of the decision variable matrix (O24:V24),
- for the third constraint, a binary matrix must be created, which determines in which source-sink pairs transportation is allowed.

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	58	19	44	54	20	57	22	58	332
F2	44	15	16	23	35	52	48	35	268
F3	28	13	46	50	17	40	14	14	222
F4	30	24	46	57	57	16	20	10	260
F5	28	34	51	21	20	40	39	24	257
F6	23	17	59	17	22	43	24	38	243
F7	43	33	51	29	30	46	26	50	308
F8	42	49	52	50	56	16	24	11	300
SUM	296	204	365	301	257	310	217	240	

	N1	N2	N3	N4	N5	N6	N7	N8
F1	0	168	30	0	100	0	34	0
F2	168	0	100	0	0	0	0	0
F3	78	0	0	0	0	0	110	34
F4	0	0	0	0	0	190	0	70
F5	0	0	0	257	0	0	0	0
F6	50	36	0	0	157	0	0	0
F7	0	0	235	0	0	0	73	0
F8	0	0	0	44	0	120	0	136

	N1	N2	N3	N4	N5	N6	N7	N8
F1	332	332	30	332	100	332	332	332
F2	268	268	100	268	268	268	268	268
F3	222	100	222	222	222	222	222	222
F4	260	260	260	260	260	260	260	120
F5	30	257	257	257	257	257	257	257
F6	50	243	243	243	243	243	243	243
F7	308	308	308	308	308	308	80	308
F8	300	300	300	300	300	120	300	300

Objective function: 54304

min	7	15	10	20	7	8	5	10
max	15	17	14	26	25	27	19	20

	N1	N2	N3	N4	N5	N6	N7	N8	min	max
F1	1	1	1	0	1	1	1	1	5	18
F2	1	0	1	0	1	1	1	1	4	12
F3	1	1	1	0	1	1	1	1	12	16
F4	0	1	0	1	1	1	1	1	15	28
F5	0	0	0	1	1	1	0	0	22	29
F6	1	1	1	0	1	1	1	1	6	20
F7	1	0	1	0	1	1	1	1	2	12
F8	1	1	1	1	1	1	1	1	6	22

	N1	N2	N3	N4	N5	N6	N7	N8	SUM
F1	0	168	30	0	100	0	34	0	332
F2	168	0	100	0	0	0	0	0	268
F3	78	0	0	0	0	0	110	34	222
F4	0	0	0	0	0	190	0	70	260
F5	0	0	0	257	0	0	0	0	257
F6	50	36	0	0	157	0	0	0	243
F7	0	0	235	0	0	0	73	0	308
F8	0	0	0	44	0	120	0	136	300
SUM	296	204	365	301	257	310	217	240	

Figure 11. Excel worksheet for solving a time- and capacity-constrained transportation problem using a linear solver (Source: own edition)

As illustrated in Fig. 11, the optimal solution obtained by the linear solver results in a total cost of 54,304 EUR.

The optimal solution of the time- and capacity-constrained transportation problem is represented as a bipartite graph in Fig. 12.

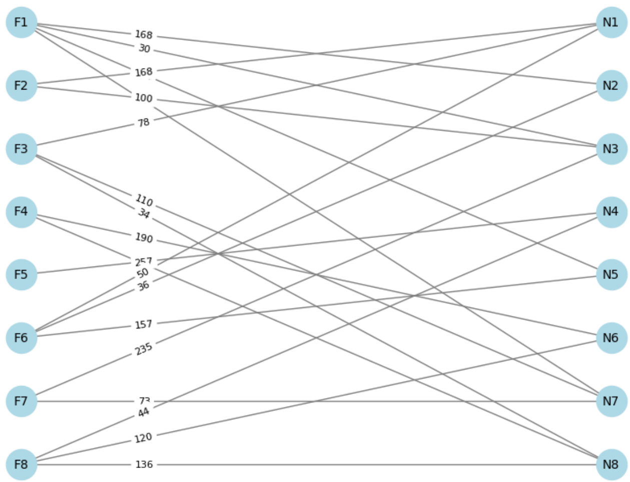

Figure 12. Optimal solution of the time- and capacity-constrained transportation problem as a bipartite graph (Source: own edition)

4. RESULTS

The results presented in Table 1 show how different constraints affect the performance and cost-efficiency of transportation models. The basic model, which includes no capacity or time restrictions, achieves the lowest total cost of 43,398 EUR with 15 deliveries. This outcome is expected, as the lack of constraints allows for maximum flexibility in assigning shipments from sources to sinks.

Table I.

Comparison of numerical results (own edition)

Modell	Objective function [EUR]	Number of routes [pcs]
Basic model	43,398	15
Capacity-constrained model	49,486	20
Time-constrained model	45,207	15
Capacity- and time- constrained model	54,304	19

Introducing capacity constraints leads to a noticeable increase in both the objective function value and the number of deliveries. Specifically, the cost rises to 49,486 EUR, and the number of deliveries increases to 20. This suggests that limiting how much can be transported between each source and sink pair forces the system to use more delivery routes, some of which are likely more expensive, thereby increasing the overall cost.

In contrast, the time-constrained model results in a moderate increase in cost, reaching 45,207 EUR, but the number of deliveries remains unchanged at 15. This implies that although time windows reduce flexibility, the system can still meet all demands using the same number of deliveries, possibly by shifting delivery timing without altering the routes significantly.

The most complex case, which includes both time and capacity constraints, yields the highest cost at 54,304 EUR and results in 19 deliveries. This highlights the cumulative effect of combining multiple restrictions, where both temporal and quantitative limits reduce the solution space considerably, requiring costlier and more diverse logistics solutions.

Overall, the comparison clearly demonstrates that additional constraints lead to higher total costs and more complex delivery structures. From a practical standpoint, decision-makers must carefully balance logistical requirements and operational constraints to optimize costs while ensuring feasibility. The trade-offs highlighted by these models underline the importance of precise planning and constraint analysis in transportation optimization problems.

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