PROBABILISTIC MODEL OF A SINGLE VEHICLE AND ONE DESTINATION ROUTING PROBLEM AND ITS MONTE CARLO SOLUTIONS

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Abstract: The paper is devoted to a shortest-time routing problem. A decision-maker has to choose one from several possible routes for a vehicle which should reach its destination as soon as possible. The time required to reach the destination depends on the chosen route and the state of traffic flow in the region and is considered as a random variable. Under an assumption that the probability distribution function of the random traffic flow depends on states of the nature which can be reasonably categorized into finite number of classes a game-theoretic approach is proposed and discussed. Then to solve the problem some global optimization procedure combined with Monte Carlo simulations is adopted. Under some additional assumptions about the route time distributions several examples are solved.

Keywords: vehicle routing problem, shortest-time criterion, global optimization, Monte Carlo simulations.

1. Introduction

The transportation industry facilitates the movement of goods for the purposes of trade, production and consumption. The transportation systems are expected to satisfy several quality factors, such as cost, time of the service and many others. To meet various expectations connected with transportations decision makers have to face many different problems in a number of different situations. An important class of the problems is a class of so called Vehicle Routing Problems (VRP). Recent results and questions arising in this field are presented and discussed in the book [5.]. Some new ideas for modelling and solving VRPs are connected with the application of the game theory methods. Such approach to various conflict and cooperative situations arising in the VRP can be found in papers [1., 3., 7., 8., 10.].

In this paper we consider a routing problem where a decision- maker has to choose one from several possible routes for a single vehicle which should reach its destination as soon as possible. We assume that the time required to reach the destination depends on the chosen route and the state of traffic flow in the region. Following the approach proposed in [6.] we formulate zero-sum matrix game against nature as a model of the problem. Next we assume that the entries of the game matrix are random variables with stochastic parameters depending on *states of nature* characterized by various types of the obstacles connected with the traffic. We also assume that, according to decision maker knowledge the states can be reasonably categorized into finite number of significantly different cases. Such a problem can be written down as a matrix game with random entries. However, in this situation an optimal solution (so called *safety strategy*) cannot be found directly with the help of the usual game theory methods. Thus in order to compute the approximation of the optimal strategy we propose to make use of a global optimization procedure based on the algorithm of the evolutionary search with soft selection. To obtain the values of the goal function for the procedure we perform Monte Carlo simulations of possible realizations of the considered random game. In the last section we present an example and address some practical questions.

2. Problems statements and their game-theoretic models

In this part of the paper we state two routing problems which are considered in the sequel. First one is the *shortest-time routing problem* the second is the *routing problem with travel time limit*.

2.1. Shortest-time routing problem (STRP)

Let v_0,v_1 denote the vertices: the depot (source) and the destination (customer), respectively. The two vertices are connected via *n* edges (routs) denoted $d_1, d_2,..,d_n$. So, we have parallel link network. The decision-maker has to choose one of the routs for the vehicle. The objective is the minimization of the transportation time, however the time required to reach the destination depends not only on the chosen route but also on the state of nature which can be one of the possible *variants* denoted $s_1, s_2,\ldots,s_m$, respectively. The estimated time necessary to reach the goal via route d_i , i=1,2,…, n, when the s_j, j=1,2,…,m, is the true state of nature is denoted T_{ij}. We assume that the decision maker does not know which state of nature *is* true (or *will be* during the travel). Such situation can be considered as a game against nature. In the game theory terminology $S=\{s_1, s_2,\ldots,s_m\}$ are called strategies for nature while the edges $D=\{d_1, d_2,..,d_n\}$ are called strategies of the decision maker. Such a game is usually denoted by $< S,D,\mathbf{T} >$, where the *loss* function **T** is given by the following *m*-by-*n* matrix

$$\begin{array}{cc} & \begin{array}{cccc} d_1 & d_2 & \ldots & d_n \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ \ldots \\ s_m \end{array} & \begin{bmatrix} T_{11} & T_{12} & \ldots & T_{1n} \\ T_{21} & T_{22} & \ldots & T_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ T_{m1} & T_{m2} & \ldots & T_{mn} \end{bmatrix} \end{array} \tag{1}$$

The possible states of nature ("strategies" of the nature) can be identified with the rows and the decision maker strategies can be identified with the columns of the above matrix **T**.

2.2. Solution concepts

Here we present briefly some fundamental solution concepts connected with the zero-sum games. They can be found in many text-books, e.g. in [9.].

In non trivial variants of the matrix **T** there are no obvious choices for the decision maker. The best choice in the case where the state of nature is s_i may be the worst when the true state is s_k and we do not know which of the states is true (or: "chosen" by the nature). The game theory provides us with various concepts of possible solutions in such a case. First concept is to use so-called minimax or safety *pure* strategies. The strategy d_{k*} (i.e. the route with the index k*) is called minimax (or safety) pure strategy if it satisfies the following condition:

$$\max_i T_{ik*} = \min_k \max_i T_{ik} \tag{2}$$

Using such a strategy the decision maker can guarantee himself that his loss does not exceed the value $V^* = \min_k \max_i T_{ik}$. The value V^* is called the *upper value of the game*.

In many situations it is possible however to propose better solution. It can be achieved by introducing so-called *mixed* strategies. From the mathematical point of view a *mixed strategy* is a probability distribution on the set D of the decision-maker pure strategies $d_1,..,d_n$. So, the mixed strategy of the decision maker is a vector $\boldsymbol{\beta} = [\beta_1, \beta_2, \dots, \beta_n]$ with nonnegative coefficients β_i, i=1,...,n, satisfying the condition:

$$\sum_{j=1}^{n} \beta_j = 1$$

Similarly we can introduce the mixed strategies for nature as vectors $\boldsymbol{\alpha}= [\alpha_1, \alpha_2, \dots,\alpha_m]$ with nonnegative α_i, i=1, ..., m, satisfying

$$\sum_{j=1}^{m} \alpha_j = 1$$

Then the expected loss $\hat{\mathbf{T}}$ for the decision maker is defined as

$$\hat{\mathbf{T}}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \cdot T_{ij} \cdot \beta_j$$

Such a game will be denoted $\mathbf{G}_1$=< $S^*,D^*,\hat{\mathbf{T}}$ >.
The minimax mixed strategy $\boldsymbol{\beta}^*$ for the game $\mathbf{G}_1$ is defined by the following condition:

$$\max_{\boldsymbol{\alpha}} \hat{\mathbf{T}}(\boldsymbol{\alpha},\boldsymbol{\beta}^*) = \min_{\boldsymbol{\beta}} \max_{\boldsymbol{\alpha}} \hat{\mathbf{T}}(\boldsymbol{\alpha},\boldsymbol{\beta}) \tag{3}$$

It appears that the upper value $v^* = \min_{\boldsymbol{\beta}} \max_{\boldsymbol{\alpha}} \hat{\mathbf{T}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ for the game < $S^*,D^*,\hat{\mathbf{T}}$ > is usually less than the upper value V^* for the game < $S,D,\mathbf{T}$ >. It is well-known that such a zero-sum games can be solved by constructing an equivalent *linear programming problem*, see e.g. [8.].

2.3. Routing problem with travel time limit (RPTTL)

In real world transportation problems we may also deal with another interesting problem. It concerns the situations where the vehicle does not have to reach its destination as soon as possible but it *cannot be late* with its cargo and that is what really matters. Such problem was considered in [7.]. To obtain the game theoretic model for such a situation we assume that a given travel time limit t^* is known to the decision maker and both the decision maker and the customer will be satisfied if the vehicle travel time is less than t^* - if so, it will be called a *success*. In such a case it would be reasonable for the decision maker to choose a strategy which maximizes the probability of the a *success*.
Such a problem can be represented by a game $\mathbf{G}_2$=< $S,D,\mathbf{T^{10}}$ > , where the matrix $\mathbf{T^{10}}$ is given as follows:

$$\mathbf{T}^{(1,0)} = \begin{bmatrix} \mathbf{1}(T_{11}) & \mathbf{1}(T_{12}) & \dots & \mathbf{1}(T_{1n}) \\ \mathbf{1}(T_{21}) & \mathbf{1}(T_{22}) & \dots & \mathbf{1}(T_{2n}) \\ \dots & \dots & \dots & \dots \\ \mathbf{1}(T_{m1}) & \mathbf{1}(T_{m2}) & \dots & \mathbf{1}(T_{mn}) \end{bmatrix} \tag{3}$$

where

$$\mathbf{1}(T_{ij}) = \begin{cases} 0 & \text{if} \quad T_{ij} > t^* \\ 1 & \text{if} \quad T_{ij} \le t^* \end{cases}$$

However, this time the entries of the game matrix can be interpreted as a *payoff* (not a *loss* as previously) for the decision maker which he wants to *maximize* - the probability of the success should be the greatest possible. So, the optimal safety strategy is now the maximin one . The strategy is defined as follows:

$$\min_{\boldsymbol{\alpha}} \mathbf{T}^{(1,0)}(\boldsymbol{\alpha},\boldsymbol{\beta}^*) = \max_{\boldsymbol{\beta}} \min_{\boldsymbol{\alpha}} \mathbf{T}^{(1,0)}(\boldsymbol{\alpha},\boldsymbol{\beta}) \tag{4}$$

3. Games with random entries

Now let us consider a more realistic case where the entries of the matrix (1) are random variables with known distribution functions. Such a model reflects the situation where a given entry of the matrix represents a random travel time connected with a given route and a given state of nature. In such a case a solution of a game cannot be obtained via linear programming methods, because we would obtain so called *chance constrained programming* model, see [2.], with (usually) a very sophisticated probabilistic nature and thus the usual methods proposed for solving such problems, see e.g. [12.], can not be adopted. One of possible ways to handle the problem is to use Monte Carlo methods. Here we propose an approach described in [6.] where the Monte Carlo methods are combined with the global optimization procedure based on evolutionary search with soft selection.

3.1. Monte Carlo simulations in finding the worst case assign to a given strategy

It results from the from the Eq. 3 that in order to find the safety strategy for STRP directly we need to compute the value $\max_{\boldsymbol{\alpha}} \hat{\mathbf{T}}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathbf{T}_*(\boldsymbol{\beta})$ for any given strategy $\boldsymbol{\beta}$. Similarly, to obtain the safety strategy for RPTTL on the base on its definition given by Eq. 4 we should have the values $\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta}) = \min_{\boldsymbol{\alpha}} \mathbf{T}^{(1,0)}(\boldsymbol{\alpha},\boldsymbol{\beta})$ for each strategy $\boldsymbol{\beta}$ of the decision maker. However, it is well-known fact that the two values can be obtained in easiest way by taking the maximum (or minimum) over the finite set of the states of nature $S=\{s_1, s_2,\ldots,s_m\}$, i.e.

$$\mathbf{T}_*(\boldsymbol{\beta}) = \max_{s_i} \hat{\mathbf{T}}(s_i,\boldsymbol{\beta}) \text{ and } \mathbf{T}_*^{(1,0)}(\boldsymbol{\beta}) = \min_{s_i} \mathbf{T}^{(1,0)}(s_i,\boldsymbol{\beta})$$

The values $\hat{\mathbf{T}}(s_i,\boldsymbol{\beta})$ and $\mathbf{T}^{(1,0)}(s_i,\boldsymbol{\beta})$ can be obtained from the formulae $\hat{\mathbf{T}}(s_i,\boldsymbol{\beta}) = \sum_{j=1}^{n} T_{ij} \cdot \beta_j$ and $\mathbf{T}^{(1,0)}(s_i,\boldsymbol{\beta}) = \sum_{j=1}^{n} T_{ij}^{(1,0)} \cdot \beta_j$ where T_{ij} and $T_{ij}^{(1,0)} = \mathbf{1}(T_{ij})$ are random variables. Let us assume that the random variables T_{ij} have known distribution. In such a case we propose to make use of the Monte Carlo method. According to the given distributions of the entries we generate N independent realizations of the matrices $\mathbf{T}$ and $\mathbf{T}^{(1,0)}$, where N is a sufficiently large number. Then we estimate (as averages) the values $\hat{\mathbf{T}}(s_i,\boldsymbol{\beta})$ and $\mathbf{T}^{(1,0)}(s_i,\boldsymbol{\beta})$ and consequently for any given strategy $\boldsymbol{\beta}$ we find the approximation of the values $\mathbf{T}_*(\boldsymbol{\beta})$ and $\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta})$ which are necessary for further optimization.

3.2. Algorithm of the evolutionary search with soft selection

First let us note that now the problem of computing the safety strategy for STRP is the problem of *minimization* of $\mathbf{T}_*(\boldsymbol{\beta})$ whilst the problem of computing the safety strategy for RPTTL is the problem of *maximization* of $\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta})$. However the first one can be equivalently treated as a problem of *maximization* of $-\mathbf{T}_*(\boldsymbol{\beta})$. The values of $-\mathbf{T}_*(\boldsymbol{\beta})$ and $\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta})$ will be called a *fitness* of the strategy $\boldsymbol{\beta}$ in the game STRP and RPTTL, respectively. Thus we can say that we need a procedure which allows us to find the global maximum of the fitness over the set of all mixed strategies of the decision maker. Here we adopt algorithm called *the evolutionary search with soft selection*, see [4., 10.]. The algorithm implemented in our simulations can be described as follows:

Step 1. Set the initial population of k vectors $\boldsymbol{\beta}_i$ ε R^n, i=1, 2, …, k (it is so called initial parent population).

Step 2. Assign to each vector $\boldsymbol{\beta}_i$,i=*1, …, k,* its fitness i.e. the value $-\mathbf{T}_*(\boldsymbol{\beta})$ or $\mathbf{T}^{(1,0)}(\boldsymbol{\beta})$, dependently on the game.

Step 3. Select parent $\boldsymbol{\beta}$ by *soft selection* i.e. with probability proportional to the its *fitness* (the better the fitness the higher the probability of the choice).

Step 4.Create a descendant $\boldsymbol{w}$ from the chosen parent $\boldsymbol{\beta}$ by its random mutation: $\boldsymbol{w}=\boldsymbol{\beta}+\mathbf{Z}$, where $\mathbf{Z}$ is a random n-dimensional vector with coefficients having expected value equal to zero and given standard deviation σ_Z.

Step 5. Repeat steps 3 and 4 for k times to create a new k-element generation of n-dimensional vectors (descendants).

Step 6. Replace the parent population with the descendant population.

Step 7. Repeat the second to sixth steps for N_G times, where N_G is a sufficiently large number.

Step 8. Return the last generation and the fitness of its elements.

The tasks of finding the safety strategies for the problems STRP and RPTTL are obviously different, so the above described procedure is realized for the two games separately.
In the next section we present some numerical examples illustrating the proposed idea.

4. Numerical examples and final remarks

Let us consider a problem where the matrix **T** is given as follows:

$$\mathbf{T} = \begin{bmatrix} 25 & 30 & 35 & 40 & 30 \\ 25 & 30 & 35 & 60 & 80 \\ 65 & 60 & 55 & 50 & 30 \\ 35 & 50 & 45 & 60 & 50 \\ 35 & 40 & 55 & 60 & 30 \\ 70 & 50 & 55 & 55 & 30 \\ 30 & 45 & 40 & 45 & 30 \\ 30 & 35 & 40 & 55 & 40 \\ 80 & 35 & 40 & 45 & 30 \end{bmatrix}$$

The matrix describes a situation when the decision maker has five different routes to choose between and he distinguishes nine significantly different states of the traffic flow which may occur during the travel and which influence the travel time. The states may be connected with the car accidents (their frequency) or some cultural or social events which are unpredictable to the decision maker and have influence on the traffic. In the matrix the travel time is given in minutes. Such a game was considered and solved in [7.]. The minimax mixed strategy is $\boldsymbol{\beta}^* = (0,0,\frac{5}{7},0,\frac{2}{7})$ and the upper value in this game equals $V^*=335/7 \cong 47{,}9$ min. The safety strategy can be interpreted as follows: the third route should be chosen with the probability 5/7, the fifth with probability 2/7 and remaining routes should be omitted.
Now let us assume that the route times are random variables and that the entries of the above matrix **T** represent the mean values of the route times in particular situations. More precisely we assume that the route times in particular case are given as $T_{ij} + Z_{ij}$ where Z_{ij} are random variables with expected values equal to zero and given finite standard deviation σ_{ij}. The quantities Z_{ij} may be interpreted as random disturbances of the average route time. Various designs of the problems are characterized by the classes the distribution of the disturbances belong to and their parameters. In our examples we analyze two distribution classes: normal distribution and translated exponential distribution. With the help of the above described simulation procedures we solve problems for various values of the standard deviation σ_{ij}. In Table 1. we present the results obtained for the normal distribution. The first column contains the values of the parameter *w* which equals the ratio σ_{ij}/T_{ij} and tell us how big is the standard deviation in comparison with the expected route time (in percents). The second column contains the fitness of the above determined strategy $\boldsymbol{\beta}^*$ which is optimal in deterministic case. The symbol $\boldsymbol{\beta}^{MM}$ denote the optimal (minimax) strategy found in our simulations for a particular design of the problem.

Table 1. Safety (minimax) strategies for STRP - normal disturbance

w	$\mathbf{T}_*(\boldsymbol{\beta}^*)$	$\boldsymbol{\beta}^{MM}$	$\mathbf{T}_*(\boldsymbol{\beta}^{MM})$
10%	47 min 54 sec	(0.00, 0.02, 0.65, 0.04, 0.29)	47 min 45 sec
30%	47 min 24 sec	(0.02, 0.08, 0.55, 0.02, 0.33)	47 min 15 sec
50%	49 min 42 sec	(0.09, 0.09, 0.51, 0.01, 0.30)	47 min 3 sec
70%	50 min 54 sec	(0.08, 0.05, 0.50, 0.00, 0.37)	45 min 42 sec

Table 2. contains the same information as Table 1. but the results were obtained for the case of translated exponential distribution.

Table 2. Safety (minimax) strategies for STRP - translated exponential disturbance

w	$\mathbf{T}_*(\boldsymbol{\beta}^*)$	$\boldsymbol{\beta}^{MM}$	$\mathbf{T}_*(\boldsymbol{\beta}^{MM})$
10%	47 min 54 sec	(0.00, 0.02, 0.65, 0.04, 0.29)	47 min 42 sec
30%	49 min 31 sec	(0.00, 0.09, 0.68, 0.00, 0.23)	47 min 14 sec
50%	48 min 20 sec	(0.00, 0.07, 0.60, 0.03, 0.30)	46 min 42 sec
70%	48 min 25 sec	(0.04, 0.01, 0.67, 0.0, 0.28)	45 min 54 sec

First of all we can notice that the strategy $\boldsymbol{\beta}^*$, optimal in deterministic case is no longer optimal when we assume the random travel times. However, it can also be seen that when the disturbances have relatively small standard deviation (w=10% or w=30%) than its performance is still quite good. It is understood, because in such cases the expected values given in the game matrix are good approximations of real route times. We can also see that the usage of the procedures based on the Monte Carlo simulations results in strategies which are always better than the strategy $\boldsymbol{\beta}^*$. The differences in performance of the strategies are bigger in the case of asymmetric distribution of disturbances, i.e. for the translated exponential one - results presented in Table 2.

Now let us consider our example in the RPTTL framework. Let us assume that a travel time limit t^* is given. It reflects the situation where the distance must be covered in a period of time that does not exceed the limit and if so, it is of no importance how many minutes it will take. Let us assume that t^*=50. The solution in the deterministic case was presented in [7.]. The safety mixed strategy in this case is $\boldsymbol{\beta}^{(1,0)} = (0, \frac{1}{2}, 0, \frac{1}{2}, 0)$. The lower value for this game is v^*=1/2 . Now let us present the results for the random route time case. They are given in Tables 3. and 4.

Table 3. Safety (maximin) strategies for RPTTL - normal disturbance

w	$\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta}^*)$	$\boldsymbol{\beta}^{(1,0)}$	$\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta}^{(1,0)})$
10%	0.28	(0.07, 0.18, 0.33, 0.00, 0.42)	0.55
30%	0.34	(0.07, 0.18, 0.26, 0.18, 0.31)	0.59
50%	0.41	(0.15, 0.10, 0.30, 0.12, 0.33)	0.59
70%	0.45	(0.02, 0.49, 0.00, 0.49, 0.00)	0.55

Table 4. Safety (maximin) strategies for RPTTL - translated exponential disturbance

w	$\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta}^*)$	$\boldsymbol{\beta}^{(1,0)}$	$\mathbf{T}_*^{(1,0)}(\boldsymbol{\beta}^{(1,0)})$
10%	0.32	(0.24, 0.00, 0.14, 0.26, 0.36)	0.50
30%	0.45	(0.05, 0.41, 0.16, 0.15, 0.23)	0.62
50%	0.52	(0.22, 0.10 0.30, 0.20, 0.18)	0.64
70%	0.59	(0.37, 0.27, 0.11, 0.15, 0.10)	0.67

The above results show again, that strategy which is optimal in deterministic case has quite poor performance in the case of random routing times. In the routing problem with travel time limit the probabilities of success may be significantly bigger - even almost two times - when we use strategies found with the help of Monte Carlo simulations.

In real world application the number of possible routes may be larger as well as the number of states of the nature. Also the distributions of routing times may belong to other distribution classes and have more sophisticated nature. However in all these situations it is possible to obtain a good solution to the routing problems with the help of proposed a approach based on global optimization procedure combined with Monte Carlo simulations.

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